

Handbook of Philosophical Logic

2nd Edition

Volume 7

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PREFACE TO THE SECOND EDITION

It is with great pleasure that we are presenting to the community the second edition of this extraordinary handbook. It has been over 15 years since the publication of the first edition and there have been great changes in the landscape of philosophical logic since then.

The first edition has proved invaluable to generations of students and researchers in formal philosophy and language, as well as to consumers of logic in many applied areas. The main logic article in the *Encyclopaedia Britannica* 1999 has described the first edition as ‘the best starting point for exploring any of the topics in logic’. We are confident that the second edition will prove to be just as good!

The first edition was the second handbook published for the logic community. It followed the North Holland one volume *Handbook of Mathematical Logic*, published in 1977, edited by the late Jon Barwise. The four volume *Handbook of Philosophical Logic*, published 1983–1989 came at a fortunate temporal junction at the evolution of logic. This was the time when logic was gaining ground in computer science and artificial intelligence circles.

These areas were under increasing commercial pressure to provide devices which help and/or replace the human in his daily activity. This pressure required the use of logic in the modelling of human activity and organisation on the one hand and to provide the theoretical basis for the computer program constructs on the other. The result was that the *Handbook of Philosophical Logic*, which covered most of the areas needed from logic for these active communities, became their bible.

The increased demand for philosophical logic from computer science and artificial intelligence and computational linguistics accelerated the development of the subject directly and indirectly. It directly pushed research forward, stimulated by the needs of applications. New logic areas became established and old areas were enriched and expanded. At the same time, it socially provided employment for generations of logicians residing in computer science, linguistics and electrical engineering departments which of course helped keep the logic community thriving. In addition to that, it so happens (perhaps not by accident) that many of the Handbook contributors became active in these application areas and took their place as time passed on, among the most famous leading figures of applied philosophical logic of our times. Today we have a handbook with a most extraordinary collection of famous people as authors!

The table below will give our readers an idea of the landscape of logic and its relation to computer science and formal language and artificial intelligence. It shows that the first edition is very close to the mark of what was needed. Two topics were not included in the first edition, even though

they were extensively discussed by all authors in a 3-day Handbook meeting. These are:

- a chapter on non-monotonic logic
- a chapter on combinatory logic and λ -calculus

We felt at the time (1979) that non-monotonic logic was not ready for a chapter yet and that combinatory logic and λ -calculus was too far removed.¹ Non-monotonic logic is now a very major area of philosophical logic, alongside default logics, labelled deductive systems, fibring logics, multi-dimensional, multimodal and substructural logics. Intensive re-examinations of fragments of classical logic have produced fresh insights, including at time decision procedures and equivalence with non-classical systems.

Perhaps the most impressive achievement of philosophical logic as arising in the past decade has been the effective negotiation of research partnerships with fallacy theory, informal logic and argumentation theory, attested to by the Amsterdam Conference in Logic and Argumentation in 1995, and the two Bonn Conferences in Practical Reasoning in 1996 and 1997.

These subjects are becoming more and more useful in agent theory and intelligent and reactive databases.

Finally, fifteen years after the start of the Handbook project, I would like to take this opportunity to put forward my current views about logic in computer science, computational linguistics and artificial intelligence. In the early 1980s the perception of the role of logic in computer science was that of a specification and reasoning tool and that of a basis for possibly neat computer languages. The computer scientist was manipulating data structures and the use of logic was one of his options.

My own view at the time was that there was an opportunity for logic to play a key role in computer science and to exchange benefits with this rich and important application area and thus enhance its own evolution. The relationship between logic and computer science was perceived as very much like the relationship of applied mathematics to physics and engineering. Applied mathematics evolves through its use as an essential tool, and so we hoped for logic. Today my view has changed. As computer science and artificial intelligence deal more and more with distributed and interactive systems, processes, concurrency, agents, causes, transitions, communication and control (to name a few), the researcher in this area is having more and more in common with the traditional philosopher who has been analysing

¹ I am really sorry, in hindsight, about the omission of the non-monotonic logic chapter. I wonder how the subject would have developed, if the AI research community had had a theoretical model, in the form of a chapter, to look at. Perhaps the area would have developed in a more streamlined way!

such questions for centuries (unrestricted by the capabilities of any hardware).

The principles governing the interaction of several processes, for example, are abstract and similar to principles governing the cooperation of two large organisations. A detailed rule based effective but rigid bureaucracy is very much similar to a complex computer program handling and manipulating data. My guess is that the principles underlying one are very much the same as those underlying the other.

I believe the day is not far away in the future when the computer scientist will wake up one morning with the realisation that he is actually a kind of formal philosopher!

The projected number of volumes for this Handbook is about 18. The subject has evolved and its areas have become interrelated to such an extent that it no longer makes sense to dedicate volumes to topics. However, the volumes do follow some natural groupings of chapters.

I would like to thank our authors and readers for their contributions and their commitment in making this Handbook a success. Thanks also to our publication administrator Mrs J. Spurr for her usual dedication and excellence and to Kluwer Academic Publishers for their continuing support for the Handbook.

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Logic	IT			
	Natural language processing	Program control specification, verification, concurrency	Artificial intelligence	Logic programming
Temporal logic	Expressive power of tense operators. Temporal indices. Separation of past from future	Expressive power for recurrent events. Specification of temporal control. Decision problems. Model checking.	Planning. Time dependent data. Event calculus. Persistence through time—the Frame Problem. Temporal query language. temporal transactions.	Extension of Horn clause with time capability. Event calculus. Temporal logic programming.
Modal logic. Multi-modal logics	generalised quantifiers	Action logic	Belief revision. Inferential databases	Negation by failure and modality
Algorithmic proof	Discourse representation. Direct computation on linguistic input	New logics. Generic theorem provers	General theory of reasoning. Non-monotonic systems	Procedural approach to logic
Non-monotonic reasoning	Resolving ambiguities. Machine translation. Document classification. Relevance theory	Loop checking. Non-monotonic decisions about loops. Faults in systems.	Intrinsic logical discipline for AI. Evolving and communicating databases	Negation by failure. Deductive databases
Probabilistic and fuzzy logic	logical analysis of language	Real time systems	Expert systems. Machine learning	Semantics for logic programs
Intuitionistic logic	Quantifiers in logic	Constructive reasoning and proof theory about specification design	Intuitionistic logic is a better logical basis than classical logic	Horn clause logic is really intuitionistic. Extension of logic programming languages
Set theory, higher-order logic, λ-calculus, types	Montague semantics. Situation semantics	Non-well-founded sets	Hereditary finite predicates	λ -calculus extension to logic programs

Imperative vs. declarative languages	Database theory	Complexity theory	Agent theory	Special comments: A look to the future
Temporal logic as a declarative programming language. The changing past in databases. The imperative future	Temporal databases and temporal transactions	Complexity questions of decision procedures of the logics involved	An essential component	Temporal systems are becoming more and more sophisticated and extensively applied
Dynamic logic	Database updates and action logic	Ditto	Possible actions	Multimodal logics are on the rise. Quantification and context becoming very active
Types. Term rewrite systems. Abstract interpretation	Abduction, relevance	Ditto	Agent's implementation rely on proof theory.	
	Inferential databases. Non-monotonic coding of databases	Ditto	Agent's reasoning is non-monotonic	A major area now. Important for formalising practical reasoning
	Fuzzy and probabilistic data	Ditto	Connection with decision theory	Major area now
Semantics for programming languages. Martin-Löf theories	Database transactions. Inductive learning	Ditto	Agents constructive reasoning	Still a major central alternative to classical logic
Semantics for programming languages. Abstract interpretation. Domain recursion theory.		Ditto		More central than ever!

Classical logic. Classical frag- ments	Basic back- ground lan- guage	Program syn- thesis	A basic tool	
Labelled deductive systems	Extremely use- ful in modelling		A unifying framework. Context theory.	Annotated logic programs
Resource and substructural logics	Lambek calcu- lus		Truth maintenance systems	
Fibering and combining logics	Dynamic syn- tax	Modules. Combining languages	Logics of space and time	Combining fea- tures
Fallacy theory				
Logical Dynamics	Widely applied here			
Argumentation theory games		Game seman- tics gaining ground		
Object level/ metalevel			Extensively used in AI	
Mechanisms: Abduction, default relevance			ditto	
Connection with neural nets				
Time-action- revision mod- els			ditto	

	Relational databases	Logical complexity classes	The workhorse of logic	The study of fragments is very active and promising.
	Labelling allows for context and control.		Essential tool.	The new unifying framework for logics
Linear logic			Agents have limited resources	
	Linked databases. Reactive databases		Agents are built up of various fibred mechanisms	The notion of self-fibring allows for self-reference
				Fallacies are really valid modes of reasoning in the right context.
			Potentially applicable	A dynamic view of logic
				On the rise in all areas of applied logic. Promises a great future
			Important feature of agents	Always central in all areas
			Very important for agents	Becoming part of the notion of a logic
				Of great importance to the future. Just starting
			A new theory of logical agent	A new kind of model

BASIC TENSE LOGIC

1 WHAT IS TENSE LOGIC?

We approach this question through an example:

- (1) *Smith*: Have you heard? Jones is going to Albania!
Smythe: He won't get in without an extra-special visa.
 Has he remembered to apply for one?
Smith: Not yet, so far as I know.
Smythe: Then he'll have to do so soon.

In this bit of dialogue the argument, such as it is, turns on issues of temporal order. In English, as in all Indo-European and many other languages, such order is expressed in part through changes in verb-form, or tenses. How should the logician treat such tensed arguments?

A solution that comes naturally to mathematical logicians, and that has been forcefully advocated in [Quine, 1960], is to regiment ordinary tensed language to make it fit the patterns of classical logic. Thus Equation 1 might be reduced to the quasi-English Equation 1 below, and thence to the 'canonical notation' of Equation 3:

- (2) Jones/visits/Albania at some time later than the present.

At any time later than the present, if Jones/visits/Albania then, then at some earlier time Jones/applies/for a visa.

At no time earlier than or equal to the present it is the case that Jones/applies/for a visa.

Therefore, Jones/applies/for a visa at some time later than the present.

- (3) $\exists t(c < t \wedge P(t))$
 $\forall t(c < t \wedge P(t) \rightarrow \exists u(u < t \wedge Q(u)))$
 $\neg \exists t((t < c \vee t = c) \wedge Q(t))$
 $\therefore \exists t(c < t \wedge Q(t)).$

Regimentation involves introducing quantification over instants $t, u, \dots$ of time, plus symbols of the present instant c and the earlier-later relation $<$. Above all, it involves treating such a linguistic item as 'Jones is visiting Albania' *not* as a complete sentence expressing a proposition and having a truth-value, to be symbolised by a sentential variable $p, q, \dots$, but rather as a predicate expressing a property on instants, to be symbolised by a one-place predicate variable $P, Q, \dots$. Regimentation has been called *detensing*

since the verb in, say, ‘Jones/visits/Albania at time t ’, written here in the grammatical present tense, ought really to be regarded as *tenseless*; for it states not a present fact but a timeless or ‘eternal’ property of the instant t . Bracketing is one convention for indicating such tenselessness. The knack for regimenting or detensing, for reducing something like Equation 1 to something like Equation 3, is easily acquired. The analysis, however, cannot stop there. For a tensed argument like that above must surely be regarded as an *enthymeme*, having as unstated premises certain assumptions about the structure of Time. Smith and Smythe, for instance, probably take it for granted that of any two distinct instants, one is earlier than the other. And if this assumption is formalised and added as an extra premise, then Equation 3, invalid as it stands, becomes valid.

Of course, it is the job of the cosmologist, not the logician, to judge whether such an assumption is physically or metaphysically correct. What *is* the logician’s job is to formalise such assumptions, correct or not, in logical symbolism. Fortunately, most assumptions people make about the structure of Time go over readily into first- or, at worst, second-order formulas.

1.1 Postulates for Earlier-Later

(B0)	Antisymmetry	$\forall x \forall y \neg (x < y \wedge y < x)$
(B1)	Transitivity	$\forall x \forall y \forall z (x < y \wedge y < z \rightarrow x < z)$
(B2)	Comparability	$\forall x \forall y (x < y \vee x = y \vee y < x)$
(B3)	(a) Maximum	$\exists x \forall y (y < x \vee y = x)$
	(b) Minimum	$\exists x \forall y (x < y \vee x = y)$
(B4)	(a) No Maximals	$\forall x \exists y (x < y)$
	(b) No Minimals	$\forall x \exists y (y < x)$
(B5)	Density	$\forall x \forall y (x < y \rightarrow \exists z (x < z \wedge z < y))$
(B6)	(a) Successors	$\forall x \exists y (x < y \wedge \neg \exists z (x < z \wedge z < y))$
	(b) Predecessors	$\forall x \exists y (y < x \wedge \neg \exists z (y < z \wedge z < x))$
(B7)	Completeness	$\forall U ((\exists x U(x) \wedge \exists x \neg U(x) \wedge$ $\forall x \forall y (U(x) \wedge$ $\wedge \neg U(y) \rightarrow x < y)) \rightarrow$ $(\exists x (U(x) \wedge$ $\wedge \forall y (x < y \rightarrow \neg U(y))) \vee$ $\exists x (\neg U(x) \wedge$ $\wedge \forall y (y < x \rightarrow U(y))))$
(B8)	Wellfoundedness	$\forall U (\exists x U(x) \rightarrow \exists x (U(x) \rightarrow$ $\wedge \forall y (y < x \rightarrow \neg U(y)))$
(B9)	(a) Upper Bounds	$\forall x \forall y \exists z (x < z \wedge y < z)$
	(b) Lower Bounds	$\forall x \forall y \exists z (z < x \wedge z < y).$

For more on the development of the logic of time as a branch of applied first- and second-order logic, see [van Benthem, 1978].

The alternative to regimentation is the development of an autonomous *tense logic* (also called *temporal logic* or *chronological logic*), first undertaken in [Prior, 1957] (though several precursors are cited in [Prior, 1967]). Tense logic takes seriously the idea that items like ‘Jones is visiting Albania’ are already complete sentences expressing propositions and having truth-values, and that they should therefore be symbolised by sentential variables $p, q, \dots$. Of course, the truth-value of a sentence in the present tense may well differ from that of the corresponding sentence in the past or future tense. Hence, tense logic will need some way of symbolising the relations between sentences that differ only in the tense of the main verb. At its simplest, tense logic adds for this purpose to classical truth-functional sentential logic just two one-place connectives: the future-tense or ‘will’ operator F and the past-tense or ‘was’ operator P . Thus, if p symbolises ‘Jones is visiting Albania’, then Fp and Pp respectively symbolise something like ‘Jones is sooner or later going to visit Albania’ and ‘Jones has at least once visited Albania’. In reading tense-logical symbolism aloud. F and P may be read respectively as ‘it will be the case that’ and ‘it was the case that’. Then $\neg F\neg$, usually abbreviated G , and $\neg P\neg$, usually abbreviated H , may be read respectively as ‘it is always going to be the case that’ and ‘it has always been the case that’. Actually, for many purposes it is preferable to take G and H as primitive, defining F and P as $\neg G\neg$ and $\neg H\neg$ respectively. Armed with this notation, the tense-logician will reduce Equation 1 above to the stylised Equation 1.1 and then to the tense-logical Equation 5:

(4) Future-tense (Jones visits Albania)

Not future-tense (Jones visits Albania and not past-tense (Jones applies for a visa)).

Not past-tense (Jones applies for a visa) and not Jones applies for a visa.

Therefore, future-tense (Jones applies for a visa)

$$\begin{aligned} (5) \quad & Fp \\ & \neg F(p \wedge \neg Pq) \\ & \neg Pq \wedge \neg q \\ & \therefore Fq. \end{aligned}$$

Of course, we will want some axioms and rules for the new temporal operators F, P, G, H . All the axiomatic systems considered in this survey will share the same standard format.

1.2 Standard Format

We start from a stock of sentential *variables* $p_0, p_2, p_2, \dots$, usually writing p for p_0 and q for p_1 . The (well-formed) *formulas* of tense logic are built

up from the variables using negation ($\neg$), and conjunction ($\wedge$), and the strong future (G) and strong past (H) operators. The *mirror image* of a formula is the result of replacing each occurrence of G by H and vice versa. Disjunction ($\vee$), material conditional ($\rightarrow$), material biconditional ($\leftrightarrow$), constant true ($\top$), constant false ($\perp$), weak future (F), and weak past (P) can be introduced as abbreviations.

As *axioms* we take all substitution instances of truth-functional tautologies. In addition, each particular system will take as axioms all substitution instances of some finite list of extra axioms, called the *characteristic* axioms of the system. As *rules* of inference we take Modus Ponens (MP) plus the specifically tense-logical:

Temporal Generalisation(TG): From α to infer $G\alpha$ and $H\alpha$

The *theses* of a system are the formulas obtainable from its axioms by these rules. A formula is *consistent* if its negation is not a thesis; a set of formulas is *consistent* if the conjunction of any finite subset is. These notions are, of course, relative to a given system.

The systems considered in this survey will have characteristic axioms drawn from the following list:

1.3 Postulates for a Past-Present-Future

(A0)	(a) $G(p \rightarrow q) \rightarrow (Gp \rightarrow Gq)$	(b) $H(p \rightarrow q) \rightarrow (Hp \rightarrow Hq)$
	(c) $p \rightarrow GPP$	(d) $p \rightarrow HFP$
(A1)	(a) $Gp \rightarrow GGp$	(b) $Hp \rightarrow HHp$
(A2)	(a) $Pp \wedge Fq \rightarrow F(p \wedge Fq) \vee F(p \wedge q) \vee F(Fp \wedge q)$	
	(b) $Pp \wedge Pq \rightarrow P(p \wedge Pq) \vee P(p \wedge q) \vee P(Pp \wedge q)$	
(A3)	(a) $G\perp \vee FG\perp$	(b) $H\perp \vee PH\perp$
(A4)	(a) $Gp \rightarrow Fp$	(b) $Hp \rightarrow Pp$
(A5)	(a) $Fp \rightarrow FFP$	(b) $Pp \rightarrow PPP$
(A6)	(a) $p \wedge Hp \rightarrow FHp$	(b) $p \wedge Gp \rightarrow PGp$
(A7)	(a) $Fp \wedge FG\neg p \rightarrow F(HFp \wedge G\neg p)$	
	(b) $Pp \wedge PH\neg p \rightarrow P(GPp \wedge H\neg p)$	
(A8)	$H(Hp \rightarrow p) \rightarrow Hp$	
(A9)	(a) $FGp \rightarrow GFp$	(b) $PHp \rightarrow HPP$

A few definitions are needed before we can state precisely the basic problem of tense logic, that of finding characteristic axioms that ‘correspond’ to various assumptions about Time.

1.4 Formal Semantics

A *frame* is a nonempty set C equipped with a binary relation R . A *valuation* in a frame (X, R) is a function V assigning each variable p_i a subset of X . Intuitively, X can be thought of as representing the set of instants of time, R

the earlier-later relation, V the function telling us *when* each p_i is the case. We extend V to a function defined on *all* formulas, by abuse of notation still called V , inductively as follows:

$$\begin{aligned} V(\neg\alpha) &= X - V(\alpha) \\ V(\alpha \wedge \beta) &= V(\alpha) \cap V(\beta) \\ V(G\alpha) &= \{x \in X : \forall y \in X (xRy \rightarrow y \in V(\alpha))\} \\ V(H\alpha) &= \{x \in X : \forall y \in X (yRx \rightarrow y \in V(\alpha))\}. \end{aligned}$$

(Some writers prefer a different notion. Thus, what we have expressed as $x \in V(\alpha)$ may appear as $\|\alpha\|_x^V = \text{TRUE}$ or as $(X, R, V) \models \alpha[x]$.) A formula α is *valid* in a frame (X, R) if $V(\alpha) = X$ for every valuation V in (X, R) , and is *satisfiable* in (X, R) if $V(\alpha) \neq \emptyset$ for some valuation V in (X, R) , or equivalently if $\neg\alpha$ is not valid in (X, R) . Further, α is *valid* over a class $\mathcal{K}$ of frames if it is valid in every $(X, R) \in \mathcal{K}$, and is *satisfiable* over $\mathcal{K}$ if it is satisfiable in some $(X, R) \in \mathcal{K}$, or equivalently if $\neg\alpha$ is not valid over $\mathcal{K}$. A system $\mathbf{L}$ in standard format is *sound* for $\mathcal{K}$ if every thesis of $\mathbf{L}$ is valid over $\mathcal{K}$, and a sound system $\mathbf{L}$ is *complete* for $\mathcal{K}$ if conversely every formula valid over $\mathcal{K}$ is a thesis of $\mathbf{L}$, or equivalently, if every formula consistent with $\mathbf{L}$ is satisfiable over $\mathcal{K}$. Any set (let us say, finite) Φ of first- or second-order axioms about the earlier-later relation $<$ determines a class $\mathcal{K}(\Phi)$ of frames, the class of its *models*. The basic correspondence problem of tense logic is, given Φ to find characteristic axioms for a system $\mathbf{L}$ that will be sound and complete for $\mathcal{K}(\Phi)$. The next two sections of this survey will be devoted to representing the solution to this problem for many important Φ .

1.5 Motivation

But first it may be well to ask, why bother? Several classes of motives for developing an autonomous tense logic may be cited:

(a) *Philosophical* motives were behind much of the pioneering work of A. N. Prior, to whom the following point seemed most important: whereas our ordinary language is tensed, the language of physics is mathematical and so untensed. Thus, there arise opportunities for confusions between different ‘terms of ideas’. Now working in tense logic, what we learn is precisely how to avoid confusing the tensed and the tenseless, and how to clarify their relations (e.g. we learn that essentially the same thought can be formulated tenselessly as, ‘Of any two distinct instants, one /is/ earlier and the other /is/ later’, and tensedly as, ‘Whatever is going to have been the case either already has been or now is or is sometime going to be the case’). Thus, the study of tense logic can have at least a ‘therapeutic’ value. Later writers have stressed other philosophical applications, and some of these are treated elsewhere in this *Handbook*.

(b) *Exegetical* applications again interested Prior (see his [Prior, 1967, Chapter 7]). Much was written about the logic of time (especially about future contingents) by such ancient writers as Aristotle and Diodoros Kronos (whose works are unfortunately lost) and by such mediaeval ones as William of Ockham or Peter Auriol. It is tempting to try to bring to bear insights from modern logic to the interpretation of their thought. But to pepper the text of an Aristotle or an Ockham with such regimenters' phrases as 'at time t ' is an almost certain guarantee of misunderstanding. For these earlier writers thought of such an item as 'Socrates is running' as being already complete as it stands, *not* as requiring supplementation before it could express a proposition or have a truth-value. Their standpoint, in other words, was like that of modern tense logic, whose notions and notations are likely to be of most use in interpreting their work, if any modern developments are.

(c) *Linguistic* motivations are behind much recent work in tense logic. A certain amount of controversy surrounds the application of tense logic to natural language. See, e.g. van Benthem [1978; 1981] for a critic's views. To avoid pointless disputes it should be emphasised from the beginning that tense logic does not attempt the faithful replication of every feature of the deep semantic structure (and still less of the surface syntax) of English or any other language; rather, it provides an idealised model giving the sympathetic linguist food for thought. an example: in tense logic, P and F can be iterated indefinitely to form, e.g. $PPPFp$ or $FPFPp$. In English, there are four types of verbal modifications indicating temporal reference, each applicable at most *once* to the main verb of a sentence: Progressive (be + ing), Perfect (have + en), Past (+ ed), and Modal auxiliaries (including will, would). Tense logic, by allowing unlimited iteration of its operators, departs from English, to be sure. But by doing so, it enables us to raise the question of whether the multiple compounds formable by such iteration are really all distinct in meaning; and a theorem of tense logic (see Section 3.5 below) tells us that on reasonable assumptions they are not, e.g. $PPPFp$ and $FPFPp$ both collapse to PFp (which is equivalent to PPp). and this may suggest *why* English does not *need* to allow unlimited iteration of its temporal verb modifications.

(d) *Computer Science*: Both tense logic itself and, even more so, the closely related so-called *dynamic logic* have recently been the objects of much investigation by theorists interested in program verification. temporal operators have been used to express such properties of programs as termination, correctness, safety, deadlock freedom, clean behaviour, data integrity, accessibility, responsiveness, and fair scheduling. These studies are mainly concerned only with *future* temporal operators, and so fall technically within the province of *modal* logic. See Harel *et al.*'s chapter on dynamic logic in Volume 4 of this *Handbook*, Pratt [1980] among other items in our bibliog-

raphy.

(e) *Mathematics*: Some taste of the purely mathematical interest of tense logic will, it is hoped, be apparent from the survey to follow. Moreover, tense logic is not an isolated subject within logic, but rather has important links with modal logic, intuitionistic logic, and (monadic) second-order logic.

Thus, the motives for investigating tense logic are many and varied.

2 FIRST STEPS IN TENSE LOGIC

Let $\mathbf{L}_0$ be the system in standard format with characteristic axioms (A0a, b, c, d). Let $\mathcal{K}_0$ be the class of *all* frames. We will show that $\mathbf{L}_0$ is (sound and) complete for $\mathcal{L}_0$, and thus deserves the title of *minimal* tense logic. The method of proof will be applied to other systems in the next section. Throughout this section, thesishood and consistency are understood relative to $\mathbf{L}_0$, validity and satisfiability relative to $\mathcal{K}_0$.

THEOREM 1 (Soundness Theorem). $\mathbf{L}_0$ is sound for $\mathcal{K}_0$.

Proof. We must show that any thesis (of $\mathbf{L}_0$) is valid (over $\mathcal{K}_0$). for this it suffices to show that each axiom is valid, and that each rule preserves validity. the verification that tautologies are valid, and that substitution and MP preserves validity is a bit tedious, but entirely routine.

To check that (A0a) is valid, we must show that for all relevant X, R, V and x , if $x \in V(G(p \rightarrow q))$ and $x \in V(Gp)$, then $x \in V(Gq)$. Well, the hypotheses here mean, first that whenever xRy and $y \in V(p)$, then $y \in V(q)$; and second that whenever xRy , then $y \in V(p)$. The desired conclusion is that whenever xRy , then $y \in V(q)$; which follows immediately. Intuitively, (A0a) says that if q is going to be the case whenever p is, and p is always going to be the case, then q is always going to be the case. The treatment of (A0b) is similar.

To check that (A0c) is valid, we must show that for all relevant X, R, V , and x , if $x \in V(p)$, then $x \in V(GPp)$. Well, the desired conclusion here is that for every y with xRy there is a z with zRy and $z \in V(p)$. It suffices to take $z = x$. Intuitively, (A0c) says that whatever is now the case is always going to have been the case. The treatment of (A0d) is similar.

To check that TG preserves validity, we must show that if for all relevant X, R, V , and x we have $x \in V(\alpha)$, then for all relevant X, R, V , and x we have $x \in V(H\alpha)$ and $x \in V(G\alpha)$, in other words, that whenever yRx we have $y \in V(\alpha)$ and whenever xRy we have $y \in V(\alpha)$. But this is immediate. Intuitively, TG says that if something is now the case *for logical reasons alone*, then for logical reasons alone it always has been and is always going to be the case: logical truth is eternal. ■

In future, verifications of soundness will be left as exercises for the reader. Our proof of the completeness of $\mathbf{L}_0$ for $\mathcal{K}_0$ will use the method of maximal

consistent sets, first developed for first-order logic by L. Henkin, systematically applied to tense logic by E. J. Lemmon and D. Scott (in notes eventually published as [Lemmon and Scott, 1977]), and refined [Gabbay, 1975].

The completeness of $\mathbf{L}_0$ for $\mathcal{K}_0$ is due to Lemmon. We need a number of preliminaries.

THEOREM 2 (Derived rules). *The following rules of inference preserve thesishood:*

1. *from $\alpha_1, \alpha_2, \dots, \alpha_n$ to infer any truth-functional consequence β*
2. *from $\alpha \rightarrow \beta$ to infer $G\alpha \rightarrow G\beta$ and $H\alpha \rightarrow H\beta$*
3. *from $\alpha \leftrightarrow \beta$ and $\theta(\alpha/p)$ to infer $\theta(\beta/p)$*
4. *from α to infer its mirror image.*

Proof.

1. To say that β is a truth-functional consequence of $\alpha_1, \alpha_2, \dots, \alpha_n$ is to say that $(\alpha_1 \wedge \alpha_2 \wedge \dots \wedge \alpha_n \rightarrow \beta)$ or equivalently $\alpha_1 \rightarrow (\alpha_2 \rightarrow (\dots (\alpha_n \rightarrow \beta) \dots))$ is an instance of a tautology, and hence is an axiom. We then apply MP.
2. From $\alpha \rightarrow \beta$ we first obtain $G(\alpha \rightarrow \beta)$ by TG, and then $G\alpha \rightarrow G\beta$ by A0a and MP. Similarly for H .
3. Here (α/p) denotes substitution of α for the variable p . It suffices to prove that if $\alpha \rightarrow \beta$ and $\beta \rightarrow \alpha$ are theses, then so are $\theta(\alpha/p) \rightarrow \theta(\beta/p)$ and $\theta(\beta/p) \rightarrow \theta(\alpha/p)$. This is proved by induction on the complexity of θ , using part (2) for the cases $\theta = G\chi$ and $\theta = H\chi$. In particular, part (3) allows us to insert and remove double negations freely. We write $\alpha \approx \beta$ to indicate that $\alpha \leftrightarrow \beta$ is a thesis.
4. This follows from the fact that the tense-logical axioms of $\mathbf{L}_0$ come in mirror-image pairs, (A0a, b) and (A0c, d). Unlike parts (1)–(3), part (4) will *not* necessarily hold for every extension of $\mathbf{L}_0$. ■

THEOREM 3 (Theses). *Items (a)–(h) below are theses of $\mathbf{L}_0$.*

Proof. We present a deduction, labelling some of the lines as theses for future reference:

(1)	$G(p \rightarrow q) \rightarrow G(\neg q \rightarrow \neg p)$	from a tautology by 1.2b
(2)	$G(\neg q \rightarrow \neg p) \rightarrow (G\neg q \rightarrow G\neg p)$	(A0a)
(a) (3)	$G(p \rightarrow q) \rightarrow (Fp \rightarrow Fq)$	from 1,2 by 1.2a
(4)	$Gp \rightarrow G(q \rightarrow p \wedge q)$	from a tautology by 1.2b
(5)	$G(q \rightarrow p \wedge q) \rightarrow (Fq \rightarrow F(p \wedge q))$	3
(b) (6)	$Gp \wedge Fq \rightarrow F(p \wedge q)$	from 4, 5 by 1.2a
(7)	$p \rightarrow GPP$	(A0c)
(8)	$GPP \wedge Fq \rightarrow F(Pp \wedge q)$	6
(c) (9)	$p \wedge Fq \rightarrow F(Pp \wedge q)$	from 7, 8 by 1.2a
(10)	$G(p \wedge q) \rightarrow Gp$	
	$G(p \wedge q) \rightarrow Gq$	from tautologies by 1.2b
(11)	$G(q \rightarrow p \wedge q) \rightarrow (Gq \rightarrow G(p \wedge q))$	(A0a)
(d) (12)	$Gp \wedge Gq \leftrightarrow G(p \wedge q)$	12
(14)	$G\neg p \wedge G\neg q \rightarrow G\neg(p \vee q)$	from 13 by 1.3c
(e) (15)	$Fp \vee Fq \leftrightarrow F(p \vee q)$	from 14 by 1.2a
(16)	$Gp \rightarrow G(p \vee q)$	
	$Gq \rightarrow G(p \vee q)$	from tautologies by 1.2b
(f) (17)	$Gp \vee Gq \rightarrow G(p \vee q)$	from 16 by 1.2a
(18)	$G\neg q \vee G\neg q \rightarrow G(\neg p \vee \neg q)$	17
(19)	$G\neg p \vee G\neg q \rightarrow G\neg(p \wedge q)$	from 18 by 1.2c
(g) (20)	$F(p \wedge q) \rightarrow Fp \wedge Fq$	from 19 by 1.2a
(21)	$\neg p \rightarrow HF\neg p$	(A0d)
(22)	$\neg p \rightarrow H\neg Gp$	from 21 by 1.2c
(h) (23)	$PGp \rightarrow p$	from 22 by 1.2a

Also the mirror images of 1.3a–h are theses by 1.2d. ■

We assume familiarity with the following:

LEMMA 4 (Lindenbaum's Lemma). *Any consistent set of formulas can be extended to a maximal consistent set.*

LEMMA 5. *Let Q be a maximal consistent set of formulas. For all formulas we have:*

1. *If $\alpha_1, \dots, \alpha_n \in A$ and $\alpha_1 \wedge \dots \wedge \alpha_n \rightarrow \beta$ is a thesis, then $\beta \in A$.*
2. *$\neg\alpha \in A$ iff $\alpha \notin A$*
3. *$(\alpha \wedge \beta) \in A$ iff $\alpha \in A$ and $\beta \in A$*
4. *$(\alpha \vee \beta) \in A$ iff $\alpha \in A$ or $\beta \in A$.*

They will be used tacitly below.

Intuitively, a maximal consistent set—henceforth abbreviated MCS—represents a full description of a possible state of affairs. For MCSs A, B we say that A is *potentially followed* by B , and write $A \dashv\vdash B$, if the conditions

of Lemma 6 below are met. Intuitively, this means that a situation of the sort described by A could be followed by one of the sort described by B .

LEMMA 6. *For any MCSs A, B , the following are equivalent:*

1. *whenever $\alpha \in A$, we have $P\alpha \in B$,*
2. *whenever $\beta \in B$, we have $F\beta \in A$,*
3. *whenever $G\gamma \in A$, we have $\gamma \in B$,*
4. *whenever $H\delta \in B$, we have $\delta \in A$.*

Proof. To show (1) implies (3): assume (1) and let $G\gamma \in A$. Then $PG\gamma \in B$, so by Thesis 3(h) we have $\gamma \in B$ as required by (3).

To show (3) implies (2): assume (3) and let $\beta \in B$. then $\neg\beta \notin B$, so $G\neg\beta \notin A$, and $F\beta = \neg G\neg\beta \in A$ as required by (2).

Similarly (2) implies (4) and (4) implies (1). ■

LEMMA 7. *Let C be an MCS, γ any formula:*

1. *if $F\gamma \in C$, then there exists an MCS B with $C \rightarrow B$ and $\gamma \in B$,*
2. *if $P\gamma \in C$, then there exists an MCS A with $A \rightarrow C$ and $\gamma \in A$.*

Proof. We treat (1): it suffices (by the criterion of Lemma 6(a)) to obtain an MCS B containing $B_0 = \{P\alpha : \alpha \in C\} \cup \{\gamma\}$. For this it suffices (by Lindenbaum's Lemma) to show that B_0 is consistent. For this it suffices (by the closure of C under conjunction plus the mirror image of Theorem 3(g)) to show that for any $\alpha \in C$, $P\alpha \wedge \gamma$ is consistent. For this it suffices (since TG guarantees that $\neg F\delta$ is a thesis whenever $\neg\delta$ is) to show that $F(P\alpha \wedge \gamma)$ is consistent. And for this it suffices to show that $F(P\alpha \wedge \gamma)$ belongs to C —as it must by 3(c). ■

DEFINITION 8. A *chronicle* on a frame (X, R) is a function T assigning each $x \in X$ an MCS $T(x)$. Intuitively, if X is thought of as representing the set of instants, and R the earlier-later relation, T should be thought of as providing a complete description of what goes on at each instant. T is *coherent* if we have $T(x) \rightarrow T(y)$ whenever xRy . T is *prophetic* (resp. *historic*) if it is coherent and satisfies the first (resp. second) condition below:

1. *whenever $F\gamma \in T(x)$ there is a y with xRy and $\gamma \in T(y)$,*
2. *whenever $P\gamma \in T(x)$ there is a y with yRx and $\gamma \in T(y)$.*

T is *perfect* if it is both prophetic and historic. Note that T is coherent iff it satisfies the two following conditions:

3. whenever $\gamma \in T(x)$ and xRy , then $\gamma \in T(y)$,
4. whenever $H\gamma \in T(x)$, and yRx , then $\gamma \in T(y)$.

If V is a valuation in (X, R) , the *induced* chronicle T_V is defined by $T_V(x) = \{\gamma : x \in V(\gamma_0)\}$; T_V is always perfect. If T is a perfect chronicle on (X, R) , the *induced* valuation V_T is defined by $V_T(p_i) = \{x : p_i \in T(x)\}$. We have:

LEMMA 9 (Chronicle Lemma). *Let T be a perfect chronicle on a frame (X, R) . If $V = V_T$ is the valuation induced by T , then $T = T_V$ the chronicle induced by V . In other words, for all formulas γ we have:*

$$(+) \quad V(\gamma) = \{x : \gamma \in T(x)\}$$

In particular, any member of any $T(x)$ is satisfiable in (X, R) .

Proof. $(+)$ is proved by induction on the complexity of γ . As a sample, we treat the induction step for G : assume $(+)$ for γ , to prove it for $G\gamma$:

On the one hand, if $G\gamma \in T(x)$, then by Definition 8(3), whenever xRy we have $\gamma \in T(y)$ and by induction hypothesis $y \in V(\gamma)$. This shows $x \in V(G\gamma)$.

On the other hand, if $G\gamma \notin T(x)$, then $F\neg\gamma \approx \neg G\gamma \in T(x)$, so by Definition 8(1) for some y with xRy we have $\neg\gamma \in T(y)$ and $\gamma \notin T(y)$, whence by induction hypothesis, $y \notin V(\gamma)$. This shows $x \notin V(G\gamma)$. ■

To prove the completeness of $\mathbf{L}_0$ for $\mathcal{K}_0$ we must show that every consistent formula γ_0 is satisfiable. Now Lemma 9 suggests an obvious strategy for proving γ_0 satisfiable, namely to construct a perfect chronicle T on some frame (X, R) containing an x_0 with $\gamma_0 \in T(x_0)$. We will construct X, R , and T piecemeal.

DEFINITION 10. Fix a denumerably infinite set W . Let M be the set of all triples (X, R, T) such that :

1. X is a nonempty finite subset of W ,
2. R is an antisymmetric binary relation on X ,
3. T is a coherent chronicle on (X, R) .

For $\mu = (X, R, T)$ and $\mu' = (X', R', T')$ in M we say μ' *extends* μ if (when relations and functions are identified with sets of ordered pairs) we have:

- 1'. $X \subseteq X'$
- 2'. $R = R' \cap (X \times X)$
- 3'. $T \subseteq T'$.

A conditional requirement of form 8(1) or (2) will be called *unborn* for $\mu = (X, R, T) \in M$ if its antecedent is not fulfilled; that is, if $x \notin X$ or if $x \in X$ but $F\gamma$ or $P\gamma$ as the case may be does not belong to $T(x)$. It will be called *alive* for μ if its antecedent is fulfilled but its consequent is not; in other words, there is no $y \in X$ with xRy or yRx as the case may be and $\gamma \in T(y)$. It will be called *dead* for μ if its consequent is fulfilled.

Perhaps no member of M is perfect; but any imperfect member of M can be improved:

LEMMA 11 (Killing Lemma). *Let $\mu = (X, R, T) \in M$. For any requirement of form 8(1) or (2) which is alive for μ , there exists an extension $\mu' = (X', R', T') \in M$ of μ for which that requirement is dead.*

Proof. We treat a requirement of form 8(1). If $x \in X$ and $F\gamma \in T(x)$, by 7(1) there is an MCS B with $T(x) \rightarrow B$ and $\gamma \in B$. It therefore suffices to fix $y \in W - X$ and set

1. $X' = X \cup \{y\}$
2. $R' = R \cup \{(x, y)\}$
3. $T' = T \cup \{(y, B)\}$. ■

THEOREM 12 (Completeness Theorem). $\mathbf{L}_0$ is complete for $\mathcal{K}_0$.

Proof. Given a consistent formula γ_0 , we wish to construct a frame (X, R) and a perfect chronicle T on it, with $\gamma_0 \in t(x_0)$ for some x_0 . To this end we fix an enumeration $x_0, x_1, x_2, \dots$ of W , and an enumeration $\gamma_0, \gamma_1, \gamma_2, \dots$ of all formulas. To the requirement of form 8(1) (resp. 8(2)) for $x = x_i$ and $\gamma = \gamma_j$ we assign the *code number* $2 \cdot 5^i \cdot 7^j$ (resp. $3 \cdot 5^i \cdot 7^j$). Fix an MCS C_0 with $\gamma_0 \in C_0$, and let $\mu_0 = (X_0, R_0, T_0)$ where $X_0 = \{x_0\}$, $R_0 = \emptyset$, and $T_0 = \{(x_0, C_0)\}$. If μ_n is defined, consider the requirement, which among all those which are alive for μ_n , has the least code number. Let μ_{n+1} be an extension of μ_n for which that requirement is dead, as provided by the Killing Lemma. Let (X, R, T) be the union of the $\mu_n = (X_n, R_n, T_n)$; more precisely, let X be the union of the X_n , R of the R_n , and T of the T_n . It is readily verified that T is a perfect chronicle on (X, R) , as required. ■

The observant reader may be wondering why in Definition 10(2) the relation R was required to be antisymmetric. the reason was to enable us to make the following remark: our proof actually shows that every thesis of $\mathbf{L}_0$ is valid over the class $\mathcal{K}_0$ of all frames, and that every formula consistent with $\mathbf{L}_0$ is satisfiable over the class $\mathcal{K}_{\text{anti}}$ of antisymmetric frames. Thus, $\mathcal{K}_0$ and $\mathcal{K}_{\text{anti}}$ give rise to the same tense logic; or to put the matter differently, there is no characteristic axiom for tense logic which ‘corresponds’ to the assumption that the earlier-later relation on instants of time is antisymmetric.

In this connection a remark is in order: suppose we let X be the set of *all* MCSs, R the relation $\rightarrow$, V the valuation $V(p_i) = \{x : p_i \in x\}$. Then using Lemmas 6 and 7 it can be checked that $V(\gamma) = \{x : \gamma \in x\}$ for *all* γ . In this way we get a quick proof of the completeness of $\mathbf{L}_0$ for $\mathcal{K}_0$. However, this (X, R) is not antisymmetric. Two MCSs A and B may be *clustered* in the sense that $A \rightarrow B$ and $B \rightarrow A$. There is a trick, known as ‘bulldozing’, though, for converting nonantisymmetric frames to antisymmetric ones, which can be used here to give an alternative proof of the completeness of $\mathbf{L}_0$ for $\mathcal{K}_{\text{anti}}$. See Bull and Segerberg’s chapter in Volume 3 of this *Handbook* and [Segerberg, 1970].

3 A QUICK TRIP THROUGH TENSE LOGIC

The material to be presented in this section was developed piecemeal in the late 1960s. In addition to persons already mentioned, R. Bull, N. Cocchiarella and S. Kripke should be cited as important contributors to this development. Since little was published at the time, it is now hard to assign credits.

3.1 Partial Orders

Let $\mathbf{L}_1$ be the extension for $\mathbf{L}_0$ obtained by adding (A1a) as an extra axiom. Let $\mathcal{K}_1$ be the class of *partial orders*, that is, of antisymmetric, transitive frames. We claim $\mathbf{L}_1$ is (sound and) complete for $\mathcal{K}_1$. Leaving the verification of soundness as an exercise for the reader, we sketch the modifications in the work of the preceding section needed to establish completeness.

First of all, we must now understand the notions of thesishood and consistency and, hence, of MCS and chronicle, as relative to $\mathbf{L}_0$. Next, we must revise clause 10(2) in the definition of M to read:

2₁. R is a partial order on X .

This necessitates a revision in clause 11(2) in the proof of the Killing Lemma. Namely, in order to guarantee that R' will be a partial order on X' , that clause must now read:

2₁. $R' = R \cup \{(x, y)\} \cup \{(v, y) : vRx\}$.

But now it must be checked that T' , as defined by clause 11(3), remains a coherent chronicle under the revised definition of R' . Namely, it must be checked that if vRx , then $T(v) \rightarrow B$. To show this (and so complete the proof) the following suffices:

LEMMA *Let A, C, B be MCSs. If $A \rightarrow C$ and $C \rightarrow B$, then $A \rightarrow B$.*

Proof. We use criterion 6(3) for $\neg\exists$: assume $G\gamma \in A$, to prove $\gamma \in B$. Well, by the new axiom (A1a) we have $GG\gamma \in A$. Then since $A \neg\exists C$, we have $G\gamma \in C$, and since $C \neg\exists B$, we have $\gamma \in B$. ■

It is worth remarking that the mirror image (A1b) of (A1a) is equally valid over partial orders, and must thus by the completeness theorem be a thesis of $\mathbf{L}_0$. To find a deduction of it is a nontrivial exercise.

3.2 Total Orders

Let $\mathbf{L}_2$ be the extension of $\mathbf{L}_1$ obtained by adding (A2a, b) as extra axioms. Let $\mathcal{K}_1$ be the class of *total orders*, or frames satisfying antisymmetry, transitivity, and comparability. Leaving the verification of soundness to the reader, we sketch the modifications in the work of Section 3.1 above, beyond simply understanding thesishood and related notions as relative to $\mathbf{L}_2$, needed to show $\mathbf{L}_2$ complete for $\mathcal{K}_2$.

To begin with, we must revise clause 10(2) in the definition of M to read:

2₂. R is a partial order on X .

This necessitates revisions in the proof of the Killing Lemma, for which the following will be useful:

LEMMA *Let A, B, C be MCSs. If $A \neg\exists B$ and $A \neg\exists C$, then either $B = C$ or $B \neg\exists C$ or $C \neg\exists B$.*

Proof. Suppose for contradiction that the two hypotheses hold but none of the three alternatives in the conclusion holds. Using criterion 6(2) for $\neg\exists$, we see that there must exist a $\gamma_0 \in C$ with $F\gamma_0 \notin b$ (else $B \neg\exists C$) and a $\beta_0 \in B$ with $F\beta_0 \notin C$ (else $C \neg\exists B$). Also there must exist a $\delta \in B, \delta \notin C$ (else $B = C$). Let $\beta = \beta_0 \wedge \neg F\gamma_0 \wedge \delta \in B, \gamma = \gamma_0 \wedge \neg F\beta_0 \wedge \neg \delta \in C$. We have $F\beta \in A$ (since $A \neg\exists B$) and $F\gamma \in A$ (since $A \neg\exists C$). hence, by A2a, one of $F(\beta \wedge F\gamma), F(F\beta \wedge \gamma), F(\beta \wedge \gamma)$ must belong to A . But this is impossible since all three are easily seen (using 3(7)) to be inconsistent. ■

Turning now to the Killing Lemma, consider a requirement of form 8(1) which is alive for a certain $\mu = (X, R, T) \in M$. We claim there is an extension $\mu' = (X', R', T')$ for which it is dead. This is proved by induction on the number n of successors which x has in (X, R) . We fix an MCS B with $T(x) \neg\exists B$ and $\gamma \in B$. If $n = 0$, it suffices to define μ' as was done in Section 3.1 above.

If $n > 0$, let x' be the immediate successor of x in (X, R) . We cannot have $\gamma \in T(x')$ or else our requirement would already be dead for μ . If $F\gamma \in T(x')$, we can reduce to the case $n - 1$ by replacing x by x' . So suppose $F\gamma \notin T(x')$. Then we have neither $B = T(x')$ nor $T(x') \neg\exists B$.

Hence, by the Lemma, we must have $B \rightarrow T(x')$. Therefore it suffices to fix $y \in W - X$ and set:

$$\begin{aligned} X' &= X \cup \{y\} \\ R' &= R \cup \{(x, y), (y, x')\} \cup \{(v, y) : vRx\} \cup \{(y, v) : (x'Rv)\} \\ I' &= T \cup \{(y, B)\}. \end{aligned}$$

In other words, we insert a point between x and x' , assigning it the set B . Requirements of form 8(2) are handled similarly, using a mirror image of the Lemma, proved using (A2b). No further modifications in the work of Section 3.1 above are called for.

The foregoing argument also establishes the following: let $\mathbf{L}_{\text{tree}}$ be the extension of $\mathbf{L}_1$ obtained by adding (A2b) as an extra axiom. Let $\mathcal{K}_{\text{tree}}$ be the class of *trees*, defined for present purposes as those partial orders in which the predecessors of any element are totally ordered. Then $\mathbf{L}_{\text{tree}}$ is complete for $\mathcal{K}_{\text{tree}}$.

It is worth remarking that the following are valid over total orders:

$$FPp \rightarrow Pp \vee p \vee Fp, \quad PFp \rightarrow Pp \vee p \vee Fp.$$

To find deductions of them in $\mathbf{L}_2$ is a nontrivial exercise. As a matter of fact, these two items could have been used instead of (A2a, b) as axioms for total orders. One could equally well have used their contrapositives:

$$Hp \wedge p \wedge Gp \rightarrow GHp, \quad Hp \wedge p \wedge Gp \rightarrow HGP.$$

The converses of these four items are valid over partial orders.

3.3 No Extremals (No Maximals, No Minimals)

Let $\mathbf{L}_3$ (resp. $\mathbf{L}_4$) be the extension of $\mathbf{L}_2$ obtained by adding (A3a, b) (resp. (A4a, b)) as extra axioms. Let $\mathcal{K}_3$ (resp. $\mathcal{K}_4$) be the class of total orders having (resp. not having) a maximum and a minimum. Beyond understanding the notions of consistency and MCS relative to $\mathbf{L}_3$ or $\mathbf{L}_4$ as the case may be, no modification in the work of Section 3.2 above is needed to prove $\mathbf{L}_3$ complete for $\mathcal{K}_3$ and $\mathbf{L}_4$ for $\mathcal{K}_4$. The following observations suffice:

On the one hand, understanding consistency and MCS relative to $\mathbf{L}_3$, if (X, R) is any total order and T any perfect chronicle on it, then for any $x \in X$, either $G\perp \in T(x)$ itself, or $FG\perp \in T(x)$ and so $G\perp \in t(y)$ for some y with xRy —this by (A3a). But if $G\perp \in T(z)$, then with w with zRw would have to have $\perp \in T(w)$, which is impossible so z must be the maximum of (X, R) . Similarly, A3b guarantees the existence of a minimum in (X, R) .

On the other hand, understanding consistency and MCS relative to $\mathbf{L}_4$, if (X, R) is any total order and T any perfect chronicle on it, then for any

$x \in X$ we have $G\top \rightarrow F\top \in T(x)$, and hence $F\top \in T(x)$, so there must be a y with $(\top \in T(y))$ and xRy —this by (A4a). Similarly, (A4b) guarantees that for any x there is a y with yRx .

The foregoing argument also establishes that the extension of $\mathbf{L}_1$ obtained by adding (A4a, b) is complete for the class of partial orders having nonmaximal or minimal elements.

It hardly needs saying that one can axiomatise the view (characteristic of Western religious cosmologies) that Time had a beginning, but will have no end, by adding (A3b) and (A4a) to $\mathbf{L}_2$.

3.4 Density

The extension $\mathbf{L}_5$ of $\mathbf{L}_2$ obtained by adding (A5a) (or equivalently (A5b)) is complete for the class $\mathcal{K}_5$ of dense total orders. The main modification in the work of Section 3.2 above needed to show this is that in addition to requirements of forms 8(1,2) we need to consider requirements of the form:

5. if xRy , then there exists a z with xRz and zRy .

To ‘kill’ such a requirement, given a coherent chronicle T on a finite total order (X, R) and $x, y \in X$ with y immediately succeeding x , we need to be able to insert a point z between x and y , and find a suitable MCS to assign to z . For this the following suffices:

LEMMA *Let A, B be MCSs with $A \rightarrow B$. Then there exists an MCS C with $A \rightarrow C$ and $C \rightarrow B$.*

Proof. The problem quickly reduces to showing $\{P\alpha : \alpha \in A\} \cup \{F\beta : \beta \in B\}$ consistent. For this it suffices to show that if $\alpha \in A$ and $\beta \in b$, then $F(P\alpha \wedge F\beta) \in A$. Now if $\beta \in B$, then since $A \rightarrow B$, $F\beta \in A$, and by (A5a), $FF\beta \in A$. An appeal to 3(3) completes the proof. ■

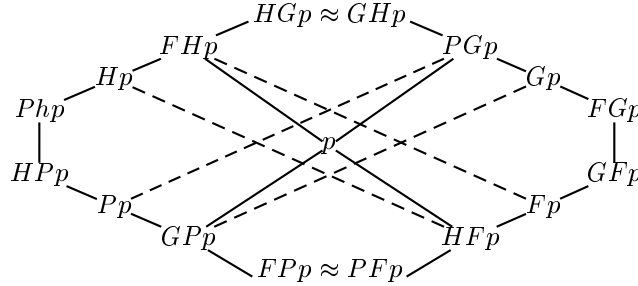


Figure 1.

Table 1.

$GGHp \approx GHp$	$FGHp \approx GHp$
$GFHp \approx GHp$	$FFHp \approx FHp$
$GPGp \approx Gp$	$FPGp \approx FGp$
$GPHp \approx PHp$	$FPHp \approx PHp$
$GFGp \approx FGp$	$FFGp \approx FGp$
$GHPp \approx HPp$	$FHPp \approx HPp$
$GGFp \approx GFp$	$FGFp \approx GFp$
$GGPp \approx GPp$	$FGPp \approx FPp$
$GHFp \approx GFp$	$FHFp \approx Fp$
$GFPp \approx FPp$	$FFPp \approx FPp$

Similarly, the extension $\mathbf{L_Q}$ of $\mathbf{L_2}$ obtained by adding (A4a, b) and (A5a) is complete for the class of dense total orders without maximum or minimum. A famous theorem tells us that any *countable* order of this class is isomorphic to the rational numbers in their usual order. Since our method of proof always produces a *countable* frame, we can conclude that $\mathbf{L_Q}$ is the tense logic of the rationals. The accompanying diagram (1) indicates some implications that are valid over dense total orders without maximum or minimum, and hence theses of $\mathbf{L_Q}$; no further implications among the formulas considered are valid. A theorem of C. L. Hamblin tells us that in $\mathbf{L_Q}$ any sequence of *Gs*, *Hs*, *Fs* and *Ps* prefixed to the variable *p* is provably equivalent to one of the 15 formulas in our diagram. It obviously suffices to prove this for sequences of length three. The reductions listed in the accompanying Table 1 together with their mirror images, suffice to prove this. It is a pleasant exercise to verify all the details.

3.5 Discreteness

The extension $\mathbf{L_6}$ of $\mathbf{L_2}$ obtained by adding (A6a, b) is complete for the class $\mathcal{K}_6$ of total orders in which every element has an immediate successor and an immediate predecessor. The proof involves quite a few modifications in the work of Section 3.2 above, beginning with:

LEMMA *For any MCS A there exists an MCS B such that:*

1. *whenever $F\gamma \in A$ then $\gamma \vee F\gamma \in B$.*

Moreover, any such MCS further satisfies:

2. *whenever $P\delta \in B$, then $\delta \vee P\delta \in A$,*

3. whenever $A \rightarrow B$, then either $B = C$ or $B \rightarrow C$,
4. whenever $C \rightarrow B$, then either $A = C$ or $C \rightarrow A$.

Proof.

1. The problem quickly reduces to proving the consistency of any finite set of formulas of the forms $P\alpha$ for $\alpha \in A$ and $\gamma \vee F\gamma$ for $F\gamma \in A$. To establish this, one notes that the following is valid over total orders, hence a thesis of $\mathbf{L}_2$ and *a fortiori* of $\mathbf{L}_6$:

$$Fp_0 \wedge Fp_1 \wedge \dots \wedge Fp_n \rightarrow \\ F((p_0 \vee Fp_0) \wedge (p_1 \vee Fp_1) \wedge \dots \wedge (p_n \vee Fp_n))$$

2. We prove the contrapositive. Suppose $\delta \vee P\delta \notin A$. By (A6a), $FH\neg\delta \in A$. by part (1), $H\neg\delta \vee FH\neg\delta \in B$. But $FHp \rightarrow Hp$ is valid over total orders, hence a thesis of $\mathbf{L}_2$ and *a fortiori* of $\mathbf{L}_6$. So $H\neg\delta \in B$ and $P\delta \notin B$ as required.
3. Assume for contradiction that $A \rightarrow B$ but neither $B = C$ nor $B \rightarrow C$. Then there exist a $\gamma_0 \in C$ with $\gamma_0 \notin B$ and a $\gamma_1 \in C$ with $F\gamma_1 \notin B$. Let $\gamma = \gamma_0 \wedge \gamma_1$. Then $\gamma \in C$ and since $A \rightarrow B$, $F\gamma \in A$. but $\gamma \vee F\gamma \notin B$, contrary to (1).
4. Similarly follows from (2). ■

We write $A \rightarrow B$ to indicate that A, B are related as in the above Lemma. Intuitively this means that a situation of the sort described by A could be *immediately* followed by one of the sort described by B .

We now take M to be the set of *quadruples* (X, R, S, T) where on the one hand, as always X is a nonempty finite subset of W , R a total order on X , and T a coherent chronicle on (X, R) ; while on the other hand, we have:

4. whenever xSy , then y immediately succeeds x in (X, R) ,
5. whenever xSy , then $T(x) \rightarrow T(y)$,

Intuitively xSy means that no points are ever to be added between x and y . We say (X', R', S', T') *extends* (X, R, S, T) if on the one hand, as always, Definition 10(1', 2', 3') hold; while on the other hand, $S \subseteq S'$. In addition to requirements of the form 8(1, 2) we need to consider requirements of the form:

5. there exists a y with xSy ,
4. there exists a y with ySx .

To ‘kill’ a requirement of form (5), take an MCS B with $T(x) \rightarrow 3'B$. If x is the maximum of (X, R) it suffices to fix $z \in W - X$ and set:

$$\begin{aligned} X' &= X \cup \{z\}, & R' &= R \cup \{(x, z)\} \cup \{(v, z) : vRx\}, \\ S' &= S \cup \{(x, z)\}, & T' &= T \cup \{(z, B)\} \end{aligned}$$

Otherwise, let y immediately succeed x in (X, R) . If $B = T(y)$ set:

$$\begin{aligned} X' &= X, & R' &= R, \\ S' &= S \cup \{(x, y)\} & T' &= T. \end{aligned}$$

Otherwise, we have $B \rightarrow 3T(y)$, and it suffices to fix $z \in W - X$ and set:

$$\begin{aligned} X' &= X, & R' &= R \cup \{(x, z), (z, y)\} \cup \\ & & & \cup \{(v, z) : vRx\} \cup \{(z, v) : yRv\}, \\ S' &= S \cup \{(x, z)\}, & T' &= T \cup \{(z, B)\} \end{aligned}$$

Similarly, to kill a requirement of form (6) we use the mirror image of the Lemma above, proved using (A6b).

It is also necessary to check that when xSy we never need to insert a point between x and y in order to kill a requirement of form 8(1) or (2). Reviewing the construction of Section 3.2 above, this follows from parts (3), (4) of the Lemma above. The remaining details are left to the reader.

A total order is *discrete* if every element but the maximum (if any) has an immediate successor, and every element but the minimum (if any) has an immediate predecessor. The foregoing argument establishes that we get a complete axiomatisation for the tense logic of discrete total orders by adding to $\mathbf{L}_2$ the following weakened versions of (A6a, b):

$$p \wedge Hp \rightarrow G \perp \vee FHp, \quad p \wedge Gp \rightarrow H \perp \vee PGp.$$

A total order is *homogeneous* if for any two of its points x, y there exists an automorphism carrying x to y . Such an order cannot have a maximum or minimum and must be either dense or discrete. In Burgess [1979] it is indicated that a complete axiomatisation of the tense logic is homogeneous orders is obtainable by adding to $\mathbf{L}_4$ the following which should be compared with (A5a) and (A6a, b):

$$(Fp \rightarrow FFp) \vee [(q \wedge Hq \rightarrow FHq) \wedge (q \wedge Gq \rightarrow PGq)].$$

3.6 Continuity

A *cut* in a total order (X, R) is a partition (Y, Z) of X into two nonempty pieces, such that whenever $y \in Y$ and $z \in Z$ we have yRz . A *gap* is a cut (Y, Z) such that Y has no maximum and Z no minimum. (X, R) is complete if it has no gaps. The *completion* (X^+, R^+) of a total order (X, R) is the complete total order obtained by inserting, for each gap (Y, Z) in (X, R) ,

an element $w(Y, Z)$ after all elements of Y and before all elements of Z . For example, the completion of the rational numbers in their usual order is the real numbers in their usual order. The extension $\mathbf{L}_7$ of $\mathbf{L}_2$ obtained by adding (A7a, b) is complete for the class $\mathcal{K}_7$ of complete total orders. The proof requires a couple of Lemmas:

LEMMA *Let T be a perfect chronicle on a total order (X, R) , and (Y, Z) a gap in (X, R) . Then if $G\alpha \in T(z)$ for all $z \in Z$, then $G\alpha \in T(y)$ for some $y \in Y$.*

Proof. Suppose for contradiction that $G\alpha \in T(z)$ for all $z \in Z$ but $F\neg\alpha \approx \neg G\alpha \in T(y)$ for all $y \in Y$. For any $y_0 \in Y$ we have $F\neg\alpha \wedge FG\alpha \in T(y_0)$. Hence, by A7a, $F(G\alpha \wedge HF\neg\alpha) \in T(y_0)$, and there is an x with y_0Rx and $G\alpha \in HF\neg\alpha \in T(x)$. But this is impossible, since if $x \in Y$ then $G\alpha \notin T(x)$, while if $x \in Z$ then $HF\neg\alpha \notin T(x)$. ■

LEMMA *Let T be a perfect chronicle on a total order (X, R) . Then T can be extended to a perfect chronicle T^+ on its completion (X^+, R^+) .*

Proof. For each gap (Y, Z) in (X, R) , the set:

$$C(Y, Z) = \{P\alpha : \exists y \in Y(\alpha \in T(y))\} \cup \{F\alpha : \exists z \in Z(\alpha \in T(z))\}$$

is consistent. This is because any finite subset, involving only $y_1, \dots, y_m$ from Y and $z_1, \dots, z_n$ from Z will be contained in $T(x)$ where x is any element of Y after all the y_i or any element of Z before all the z_j . Hence, we can define a coherent chronicle T^+ on (X^+, R^+) by taking $T^+(w(Y, Z))$ to be some MCS extending $C(Y, Z)$. Now if $F\alpha \in T^+(w(Y, Z))$, we claim that $F\alpha \in T(z)$ for some $z \in Z$. For if not, then $G\neg\alpha \in T(z)$ for all $z \in Z$, and by the previous Lemma, $G\neg\alpha \in T(y)$ for some $y \in Y$. But then $PG\neg\alpha$, which implies $\neg F\alpha$, would belong to $C(Y, Z) \subseteq T^+(w(Y, Z))$, a contradiction. It hardly needs saying that if $F\alpha \in T(z)$, then there is some x with zRx and *a fortiori* $w(Y, Z)R^+x$ having $\alpha \in T(x)$. This shows T^+ is prophetic. Axiom (A7b) gives us a mirror image to the previous Lemma, which can be used to show T^+ historic. ■

To prove the completeness of $\mathbf{L}_7$ for $\mathcal{K}_7$, given a consistent γ_0 use the work of Section 2.2 above to construct a perfect chronicle T on a frame (X, R) such that $\gamma_0 \in T(x_0)$ for some x_0 . Then use the foregoing Lemma to extend to a perfect chronicle on a complete total order, as required to prove satisfiability. ■

Similarly, $\mathbf{L}_R$, the extension of $\mathbf{L}_2$ obtained by adding (A4a, b) and (A5a) and (A7a, b) is complete for the class of complete dense total orders without maximum or minimum, sometimes called *continuous* orders. As a matter of fact, our construction shows that any formula consistent with this theory is satisfiable in the completion of the rationals, that is, in the reals. Thus $\mathbf{L}_R$ is the tense logic of real time and, hence, of the time of classical physics.

3.7 Well-Orders

The extension $\mathbf{L}_8$ of $\mathbf{L}_2$ obtained by adding (A8) is complete for the class $\mathcal{K}_8$ of all well-orders. For the proof it is convenient to introduce the abbreviations Ip for $Pp \vee p \vee Fp$ or ‘ p sometime’, and Bp for $p \wedge \neg Pp$ or ‘ p for the first time’. an easy consequence of (A8) is $Ip \rightarrow IBp$: if something *ever* happens, then there is a *first* time when it happens the reader can check that the following are valid over total orders; hence, theses of ($\mathbf{L}_2$ and *a fortiori* of $\mathbf{L}_9$):

1. $Ip \wedge Iq \rightarrow I(Pp \wedge q) \vee I(p \wedge q) \vee I(p \wedge Pq)$,
2. $I(q \wedge Fr) \wedge I(PBp \wedge Bq) \rightarrow I(p \wedge Fr)$.

Now, understanding consistency, MCS, and related notions relative to $\mathbf{L}_8$, let δ_0 be any consistent formula and D_0 any MCS containing it. Let $\delta_1, \dots, \delta_k$ be all the proper subformulas of δ_0 . Let Γ be the set of formulas of form

$$(\neg)\delta_0 \wedge (\neg)\delta_1 \wedge \dots \wedge (\neg)\delta_k$$

where each δ_i appears once, plain or negated. Note that distinct elements of Γ are truth-functionally inconsistent. Let $\Gamma' = \{\gamma \in \Gamma : I\gamma \in D_0\}$. Note that for each $\gamma \in \Gamma'$ we have $IB\gamma \in D_0$, and that for distinct $\gamma, \gamma' \in \Gamma'$ we must by (1) have either $I(PB\gamma \wedge B\gamma')$ or $I(PB\gamma' \wedge B\gamma)$ in D_0 . Enumerate the elements of Γ' as $\gamma_0, \gamma_1, \dots, \gamma_n$ so that $I(PB\gamma_i \wedge B\gamma_j) \in D_0$ iff $i < j$. We write $i \triangleleft j$ if $I(\gamma_i \wedge F\gamma_j) \in D_0$. This clearly holds whenever $i < j$, but may also hold in other cases. A crucial observation is:

$$(+) \quad \text{If } i < j \leq k \text{ and } k \triangleleft i, \text{ then } j \triangleleft i$$

This follows from (2). These tedious preliminaries out of the way, we will now define a set X of ordinals and a function t from X to Γ' . Let $a, b, c, \dots$ range over *positive* integers:

We put $0 \in X$ and set $t(0) = \gamma_0$.

If $0 \triangleleft 0$ we also put each $a \in X$ and set $t(a) = \gamma_0$.

We put $\omega \in X$ and set $t(\omega) = \gamma_1$.

If $1 \triangleleft 1$ we also put each $\xi = \omega \cdot b \in X$ and set $t(\xi) = \gamma_1$.

If $1 \triangleleft 0$ we also put each $\xi = \omega \cdot b + a \in X$ and set $t(\xi) = \gamma_0$.

We put $\omega^2 \in X$ and set $t(\omega^2) = \gamma_2$.

If $2 \triangleleft 2$ we also put each $\xi = \omega^2 \cdot c \in X$ and set $t(\xi) = \gamma_2$.

If $2 \triangleleft 1$ we also put each $\xi = \omega^2 \cdot c + \omega \cdot b \in X$, and set $t(\xi) = \gamma_1$.

If $2 \triangleleft 0$ we also put each $\xi = \omega^2 \cdot c + \omega \cdot b + a \in X$ and set $t(\xi) = \gamma_0$.

and so on.

Using (+) one sees that whenever $\xi, \eta \in X$ and $\xi < \eta$, then $i \triangleleft j$ where $t(\xi) = \gamma_i$ and $t(\eta) = \gamma_j$. Conversely, inspection of the construction shows that:

1. whenever $\xi \in X$ and $t(\xi) = \gamma_j$ and $j \triangleleft k$, then there is an $\eta \in X$ with $\xi < \eta$ and $t(\eta) = \gamma_k$
2. whenever $\xi \in X$ and $t(\xi) = \gamma_j$ and $i < j$, then there is an $\eta \in X$ with $\eta < \xi$ and $t(\eta) = \gamma_i$.

For $\xi \in X$ let $T(\xi)$ be the set of conjuncts of $t(\xi)$. Using (1) and (2) one sees that T satisfies all the requirements 8(1,2,3,4) for a perfect chronicle, *so far as these pertain to subformulas of δ_0* . Inspection of the proof of Lemma 9 then shows that this suffices to prove δ_0 satisfiable in the well-order $(X, <)$. ■

Without entering into details here, we remark that variants of $\mathbf{L}_8$ provide axiomatisations of the tense logics of the integers, the natural numbers, and of finite total orders. In particular, for the natural numbers one uses $\mathbf{L}_\omega$, the extension of $\mathbf{L}_2$ obtained by adding (A8) and $p \wedge Gp \rightarrow H\perp \vee PGp$. $\mathbf{L}_\omega$ is the tense logic of the notion of time appropriate for discussing the working of a digital computer, or of the mental mathematical constructions of Brouwer's 'creative subject'.

3.8 Lattices

The extension $\mathbf{L}_9$ of $\mathbf{L}_1$ obtained by adding (A4a, b) and (A9a, b) is complete for the class $\mathcal{K}_9$ of partial orders without maximal or minimal elements in which any two elements have an upper and a lower bound. We sketch the modifications in the work of Section 3.2 above needed to prove this:

To begin with, we must revise clause 10(2) in the definition of M to read:

- 2g. R is a partial order on X having a maximum and a minimum.

This necessitates revisions in the proof of the Killing Lemma, for which the following will be useful:

LEMMA *Let A, B, C be MCSs. If $A \rightarrow B$ and $A \rightarrow C$, then there exists an MCS D such that $B \rightarrow D$ and $C \rightarrow D$.*

Proof. The problem quickly reduces to showing $\{\beta : G\beta \in B\} \cup \{\gamma : G\gamma \in C\}$ consistent. For this it suffices (using 3(4)) to show that $\beta \wedge \gamma$ is consistent whenever $G\beta \in B, G\gamma \in C$. Now in that case we have $FG\beta, FG\gamma \in A$, since $A \rightarrow B, C$. By A9a, we then have $GF\beta \in A$, and by 3(2) we then have $F(F\beta \wedge G\gamma) \in A$ and $FF(\beta \wedge \gamma) \in A$, which suffices to prove $\beta \wedge \gamma$ consistent as required. ■

Turning now to the Killing Lemma, trouble arises when for a given $(X, R, T) \in M$ a requirement of form Definition 8(1) is said to be 'killed' for some x other than the maximum y of (X, R) and some $F\gamma \in T(x)$.

Fixing an MCS B with $T(x) \rightarrow B$ and $\gamma \in B$, and $az \in W - X$, we would like to add z to x placing it after x and assigning it the MCS B . But we cannot simply do this, else the resulting partial order would have no maximum. (For y and z would be incomparable.) So we apply the Lemma (with $A = T(x), C = T(y)$) to obtain an MCS D with $B \rightarrow D$ and $T(y) \rightarrow D$. We fix a $w \in W - X$ distinct from z , and set:

$$\begin{aligned} X' &= X \cup \{z, w\}, \\ R' &= R \cup \{(x, z), (z, w)\} \cup \{(v, z) : vRx\} \cup \{(v, w) : v \in X\}. \\ T' &= T \cup \{(z, B), (w, D)\}. \end{aligned}$$

Similarly, a requirement of form 8(2) involving an element other than the minimum is treated using the mirror image of the Lemma above, proved using (A9b).

Now given a formula γ_0 consistent with $\mathbf{L}_9$, the construction of Definition 10 above produces a perfect chronicle T on a partial order (X, R) with $\gamma_0 \in T(x_0)$ for some x_0 . The work of Section 2.4 above shows that (X, R) will have no maximal or minimal elements. Moreover, (X, R) will be a union of partial orders (X_n, R_n) satisfying (2g). Then any $x, y \in X$ will have an R -upper bound and an R -lower bound, namely the R_n -maximum and R_n -minimum elements of any X_n containing them both. Thus, $(X, R) \in \mathcal{K}_9$ and γ_0 is satisfiable over $\mathcal{K}_9$. ■

A *lattice* is a partial order in which any two elements have a *least* upper bound and a *greatest* lower bound. Actually, our proof shows that $\mathbf{L}_9$ is complete for the class of lattices without maximum or minimum. It is worth mentioning that (A9a, b) could have been replaced by:

$$Fp \wedge Fq \rightarrow F(Pp \wedge Pq), \quad Pp \wedge Pq \rightarrow P(Fp \wedge Fq).$$

Weakened versions of these axioms can be used to give an axiomatisation for the tense logic of arbitrary lattices.

4 THE DECIDABILITY OF TENSE LOGICS

All the systems of tense logic we have considered so far are recursively decidable. Rather than give an exhaustive (and exhausting) survey, we treat here two examples, illustrating the two basic methods of proving decidability: one method, borrowed from modal logic, is that of using so-called *filtrations* to establish what is known as the *finite model property*. The other, borrowed from model theory, is that of using so-called *interpretations* in order to be able to exploit a powerful theorem of [Rabin, 1966].

THEOREM 13. $\mathbf{L}_9$ is decidable.

Proof. Let $\mathcal{K}$ be the class of models of (B1) and (B9a, b); thus $\mathcal{K}$ is like $\mathcal{K}_9$ except that we do *not* require antisymmetry. Let $\mathcal{K}'$ be the class of finite

elements of $\mathcal{K}$. It is readily verified that $\mathbf{L}_9$ is sound for $\mathcal{K}$ and *a fortiori* for $\mathcal{K}'$. We claim that $\mathbf{L}_9$ is complete for $\mathcal{K}'$. This provides an effective procedure for testing whether a given formula α is a thesis of $\mathbf{L}_9$ or not, as follows: search simultaneously through all deductions in the system $\mathbf{L}_9$ and through all members of $\mathcal{K}'$ —or more precisely, of some nice countable subclass of $\mathcal{K}'$ containing at least one representative of each isomorphism-type. Eventually one either finds a deduction of α , in which case α is a thesis, or one finds an element of $\mathcal{K}'$ in which $\neg\alpha$ is satisfiable, in which case by our completeness claim, α is not a thesis.

To prove our completeness claim, let γ_0 be consistent with $\mathbf{L}_9$. We showed in Section 2.9 above how to construct a perfect chronicle T on a frame $(X, R) \in \mathcal{K}_9 \subseteq \mathcal{K}$ having $\gamma_0 \in T(x_0)$ for some x_0 . For $x \in X$, let $t(x)$ be the set of subformulas of γ_0 in $T(x)$. Define an equivalence relation on X by:

$$x \leftrightarrow y \text{ iff } t(x) = t(y).$$

Let $[x]$ denote the equivalence class of x , X' the set of all $[x]$. Note that X' is finite, having no more than 2^k elements, where k is the number of subformulas of γ_0 . Consider the relations on X' defined by:

$$\begin{aligned} aR^+b & \text{ iff } xRy \text{ for some } x \in a \text{ and } y \in b, \\ aR'b & \text{ iff } \text{for some finite sequence } a = c_0, c_1, \dots, c_{n-1}, c_n = b \\ & \text{ we have } c_i R^+ c_{i+1} \text{ for all } i < n. \end{aligned}$$

Clearly R' is transitive, while R^+ and, hence, R' inherit from R the properties expressed by B9a, b. Thus $(X', R') \in \mathcal{K}'$. Define a function t' on X' by letting $t'(a)$ be the common value of $t(x)$ for all $x \in a$. In particular for $a_0 = [x_0]$ we have $\gamma_0 \in t'(a_0)$. We claim that t' satisfies clauses 8(1, 2, 3, 4) of the definition of a perfect chronicle *so far as these pertain to subformulas of γ_0* . As remarked in Section 3.8 above, this suffices to show γ_0 satisfiable in (X', R') and, hence, satisfiable over $\mathcal{K}'$ as required.

In connection with Definition 8(1), what we must show is:

1. whenever $F\gamma \in t(a)$ there is a b with $aR'b$ and $\gamma \in t(b)$

Well, let $a = [x]$, so $F\gamma \in t(x) \subseteq T(x)$. There is a y with xRy and $\gamma \in t(y)$ since T is prophetic. Letting $b = [y]$ we have aR^+b and so $aR'b$.

In connection with Definition 8(3) what we must show is:

- 3'. whenever $G\gamma \in t(a)$ and $aR'b$, then $\gamma \in t(b)$.

For this it clearly suffices to show:

- 3⁺ whenever $G\gamma \in t(a)$ and aR^+b , then $\gamma \in t(b)$ and $G\gamma \in t(b)$.

To show this, assuming the two hypotheses, fix $x \in a$ and $y \in b$ with xRy . We have $G\gamma \in t(x) \subseteq T(x)$, so by (A1a), $GG\gamma \in T(x)$. Hence, $\gamma \in t(y)$ and $G\gamma \in t(y)$, since T is coherent—which completes the proof.

Definitions 8(2, 4) are treated similarly. ■

THEOREM 14. $\mathbf{L}_R$ is decidable.

Proof. We introduce an alternative definition of *validity* which is useful in other contexts. To each tense-logical formula α we associate a first-order formula $\hat{\alpha}$ as follows: for a sentential variable p_i we set $\hat{p}_i = P_i(x)$ where P_i is a one-place predicate variable. We then proceed inductively:

$$\begin{aligned} (\neg\alpha)^\wedge &= \neg\hat{\alpha}, \\ (\alpha \wedge \beta)^\wedge &= \hat{\alpha} \wedge \hat{\beta} \\ (G\alpha)^\wedge &= \forall y(x < y \rightarrow \hat{\alpha}(y/x)), \\ (H\alpha)^\wedge &= \forall y(y < x \rightarrow \hat{\alpha}(y/x)). \end{aligned}$$

Here (y/x) represents the result of substituting for x the alphabetically first variable y not occurring yet. Given a valuation V in a frame (X, R) we have an interpretation in the sense of first-order model theory, in which R interprets the symbol $<$ and $V(p_i)$ the symbol P_i . Unpacking the definitions it is entirely trivial that we always have:

$$(*) \quad a \in V(\alpha) \text{ iff } (X, R, V(p_0), V(p_1), V(p_2), \dots) \models \hat{\alpha}(x),$$

where $\models$ is the usual satisfaction relation of model theory. We now further define:

$$a^+ = \forall P_0 \forall P_1, \dots, \forall P_k \forall x \hat{\alpha}(x),$$

where $p_0, p_1, \dots, p_k$ include all the variables occurring in α . Note that α^+ is a second-order formula of the simplest kind: it is *monadic* (all its second-order variables are *one*-place predicate variables) and *universal* (consisting of a string of universally-quantified second-order variables prefixed to a first-order formula). It is entirely trivial that:

$$(+)\quad \alpha \text{ is valid in } (X, R) \text{ iff } (X, R) \models \alpha^+$$

It follows that to prove the decidability of the tense logic of a given class $\mathcal{K}$ of frames it will suffice to prove the decidability of the set of universal monadic (second-order) formulas true in all members of $\mathcal{K}$.

Let $2^{<\omega}$ be the set of all finite 0,1-sequences. Let $*0$ be the function assigning the argument $s = (i_0, i_1, \dots, i_n) \in 2^{<\omega}$ the value $s * 0 = (i_0, i_1, \dots, i_n, 0)$, and similarly for $*1$. Rabin proves the decidability of the set $S2S$ of monadic (second order) formulas true in the structure $(2^{<\omega}, *0, *1)$. He deduces as an easy corollary the decidability of the set of monadic formulas true in the frame $(\mathbb{Q}, <)$ consisting of the rational numbers with their usual order. This immediately yields the decidability of the system $\mathbf{L}_Q$ of Section 2.5 above. Further corollaries relevant to tense logic are the decidability of the set of monadic formulas true in all countable total orders, and similarly for countable well-orders.

It only remains to reduce the decision problem for $\mathbf{L}_R$ to that for $\mathbf{L}_Q$. The work of 2.7 above shows that a formula α is satisfiable in the frame $(\mathbb{R}, <)$ consisting of the real numbers with their usual order, iff it is satisfiable in the frame $(\mathbb{Q}, <)$ by a valuation V with the property:

1. $V(\alpha) = \mathbb{Q}$ for every substitution instance α of (A7a or b).

Inspection of the proof actually shows that it suffices to have:

2. $V(\alpha') = \mathbb{Q}$ where α' is the conjunction of all instances of (A7a or b) obtainable by substituting subformulas of α for variables.

A little thought shows that this amounts to demanding:

3. $V(\alpha \wedge GH\alpha') \neq \emptyset$.

In other words, α is satisfiable in $(\mathbb{R}, <)$ iff $\alpha \wedge GH\alpha'$ is satisfiable in $(\mathbb{R}, <)$, which effects the desired reduction. For the lengthy original proof see [Bull, 1968]. Other applications of Rabin's theorem are in [Gabbay, 1975]. Rabin's proof uses automata-theoretic methods of Büchi; these are avoided by [Shelah, 1975]. ■

5 TEMPORAL CONJUNCTIONS AND ADVERBS

5.1 *Since, Until, Uninterruptedly, Recently, Soon*

All the systems discussed so far have been based on the primitives $\neg, \wedge, G, H$. It is well-known that any truth function can be defined in terms of $\neg, \wedge$. Can we say something comparable about temporal operators and G, H ? When this question is formulated precisely, the answer is a resounding NO.

DEFINITION 15. Let φ be a first-order formula having one free variable x and no nonlogical symbols but the two-place predicate $<$ and the one-place predicates $P_1, \dots, P_n$. corresponding to φ we introduce a new n -place connective, the (first-order, one-dimensional) temporal operator $O(\varphi)$. We describe the formal semantics of $O(\varphi)$ in terms of the alternative approach of Theorem 14 above: we add to the definition of $\hat{}$ the clause:

$$(O(\varphi)(\alpha_1, \dots, \alpha_n))^{\hat{}} = \varphi(\hat{\alpha}_1/P_1, \dots, \hat{\alpha}_n/P_n).$$

Here $\hat{\alpha}/P$ denotes substitution of the formula $\hat{\alpha}$ for the predicate variable P . We then let formula (*) of Theorem 14 above *define* $V(\alpha)$ for formulas α involving $O(\varphi)$. Examples 16 below illustrate this rather involved definition. If $\mathcal{O} = \{O(\varphi_1), \dots, O(\varphi_k)\}$ is a set of temporal operators, an $\mathcal{O}$ -*formula* is one built up from sentential variables using $\neg, \wedge$, and elements of $\mathcal{O}$. A temporal operator $O(\varphi)$ is $\mathcal{O}$ -*definable* over a class $\mathcal{K}$ of frames if there is an $\mathcal{O}$ -formula α such that $O(\varphi)(p_1, \dots, p_n) \leftrightarrow \alpha$ is valid over $\mathcal{K}$. $\mathcal{O}$ is

temporally complete over $\mathcal{K}$ if every temporal operator is $\mathcal{O}$ -definable over $\mathcal{K}$. Note that the smaller $\mathcal{K}$ is—it may consist of a single frame—the easier it is to be temporally complete over it.

EXAMPLES 16.

1. $\forall y(x < y \rightarrow P_1(y))$
2. $\forall y(y < x \rightarrow P_1(y))$
3. $\exists y(x < y \wedge \forall z(x < z \wedge z < y \rightarrow P_1(z)))$
4. $\exists y(y < x \wedge \forall z(y < z \wedge z < x \rightarrow P_1(z)))$
5. $\exists y(x < y \wedge P_1(y) \wedge \forall z(y < z \wedge z < x \rightarrow P_1(z)))$

For (1), $O(\varphi)$ is just G . For (2), $O(\varphi)$ is just H . For (3), $O(\varphi)$ will be written G' , and may be read ‘ p is going to be uninterruptedly the case for some time’. For (4), $O(\varphi)$ will be written H' , and may be read ‘ p has been uninterruptedly the case for some time’. For (5), $O(\varphi)$ will be written U , and $U(p, q)$ may be read ‘until p, q ’; it predicts a future occasion of p ’s being the case, up until which q is going to be uninterruptedly the case. For (6), $O(\varphi)$ will be written S , and $S(p, q)$ may be read ‘since p, q ’. In terms of G' we define $F' = \neg G; \neg$, read ‘ p is going to be the case arbitrarily soon’. In terms of H' we define $P' = \neg H' \neg$, read ‘ p has been the case arbitrarily recently’. Over all frames, Gp is definable as $\neg U(\neg p, \top)$, and G' as $U(\top, p)$. Similarly, H and H' are definable in terms of S . The following examples are due to H. Kamp:

PROPOSITION 17. G' is not G, H -definable over the frame $(\mathbb{R}, <)$.

Sketch of Proof. Define two valuations over that frame by:

$$V(p) = \{0, \pm 1, \pm 2, \pm 3, \dots\} \quad W(p) = V(p) \cup \{\pm \frac{1}{2}, \pm \frac{1}{4}, \pm \frac{1}{8}, \dots\}$$

Then intuitively it is plausible, and formally it can be proved that for any G, H -formula α we have $0 \in V(\alpha)$ iff $0 \in W(\alpha)$. But $0 \in V(G'p) - W(G'p)$. ■

PROPOSITION 18. U is not G, H, G', H' -definable over the frame $(\mathbb{R}, <)$.

Sketch of Proof. Define two valuations by:

$$\begin{aligned} V(p) &= \{\pm 1, \pm 2, \pm 3, \pm 4, \dots\} & W(p) &= \{\pm 2, \pm 3, \pm 4, \dots\} \\ V(q) &= W(q) = \text{the union of the open intervals} \\ &\quad \dots, (-5, -4), (-3, -2), (-1, +1), \\ &\quad (+2, +3), (+4, +5), \dots \end{aligned}$$

Then intuitively it is plausible, and formally it can be proved that for any G, H, G', H' -formula α we have $0 \in V(\alpha)$ iff $0 \in W(\alpha)$. But $0 \in V(U(p, q) - W(U(p, q)))$. ■

Such examples might inspire pessimism, but [Kamp, 1968] proves:

THEOREM 19. *The set $\{U, S\}$ is temporally complete over continuous orders.*

We will do no more than outline the difficult proof (in an improved version due to Gabbay): Let $\mathcal{O}$ be a set of temporal operators, $\mathcal{K}$ a class of frames. An $\mathcal{O}$ -formula α is *purely past* over $\mathcal{K}$ if whenever $(X, R) \in \mathcal{K}$ and $x \in K$ and V, W are valuations in (X, R) agreeing before x (so that for all i , $V(p_i) \cap \{y : yRx\} = W(p_i) \cap \{y : yRx\}$) then $x \in V(\alpha)$ iff $x \in W(\alpha)$. Similarly, one defines *purely present* and *purely future*, and one defines *pure* to mean purely past, or present, or future. Note that $Hp, H'p, S(p, q)$, are purely past, their mirror images purely future, and any truth-functional compound of variables purely present. $\mathcal{O}$ has the *separation property* over $\mathcal{K}$ if for every $\mathcal{O}$ -formula α there exists a truth-functional compound β of $\mathcal{O}$ -formulas pure over $\mathcal{K}$ such that $\alpha \leftrightarrow \beta$ is valid over $\mathcal{K}$. $\mathcal{O}$ is *strong* over $\mathcal{K}$ if G, H are $\mathcal{O}$ -definable over $\mathcal{K}$. Gabbay [1981a] proves:

Criterion 20. *Over any given class $\mathcal{K}$ of total orders, if $\mathcal{O}$ is strong and has the separation property, then it is temporally complete.*

A full proof being beyond the scope of this survey (see, however, the next chapter ‘Advanced Tense Logic’), we offer a sketch: we wish to find for any first-order formula $\varphi(x, <, P_1, \dots, P_n)$ an $\mathcal{O}$ -formula $\alpha(p_1, \dots, p_n)$ *representing* it in the sense that for any $(X, R) \in \mathcal{K}$ and any valuation V and any $a \in X$ we have:

$$a \in V(\alpha) \text{ iff } (X, R, V(p_1), \dots, V(p_n)) \models \varphi(a/x).$$

The proof proceeds by induction on the depth of nesting of quantifiers in φ , the key step being $\varphi(x) = \exists y \psi(x, y)$. In this case, the atomic subformulas of ψ are of the forms $P_i(x), P_i(z), z < x, z = x, x < z, z = w, z < w$, where z and w are variables other than x . Actually, we may assume there are no subformulas of the form $P_i(x)$ since these can be brought outside the quantifier $\exists y$. We introduce new singulary predicates Q^-, Q^0, Q^+ and replace the subformulas of ψ of forms $z < x, z = x, x < z$ by $Q^-(z), Q^0(z), Q^+(z)$, to obtain a formula $\vartheta(y, <, P_1, \dots, P_n, Q^-, Q^0, Q^+)$ to which we can apply our induction hypothesis, obtaining an $\mathcal{O}$ -formula $\delta(p_1, \dots, p_n, q^-, q^0, q^+)$ representing it. Let $\gamma(p_1, \dots, sp_n) = \delta(p-1, \dots, p_n, Fq, q, Pq)$, and $\beta = P\gamma \vee \gamma \vee F\gamma$. It is readily verified that for any $(X, R) \in \mathcal{K}$ and any $a, b \in X$ and any valuation V with $V(q) = \{a\}$ that we have:

$$\begin{aligned} b \in V(\gamma) & \text{ iff } (X, R, V(p_1), \dots, V(p_n)) \models \psi(a/x, b/y), \\ a \in V(\beta) & \text{ iff } (X, R, V(p_1), \dots, V(p_n)) \models \varphi(a/x). \end{aligned}$$

By hypothesis, β is equivalent over $\mathcal{K}$ to a truth-functional compound of purely past formulas β_i^- , purely present ones β_j^0 , and purely future ones

β_k^+ . In each β_i^- (resp. β_j^0) (resp. β_k^+) replace q by $\perp$ (resp. $\top$) (resp. $\perp$) to obtain an $\mathcal{O}$ -formula α . It is readily verified that α represents φ .

It ‘only’ remains to show:

LEMMA 21. *The set $\{U, S\}$ has the separation property over complete orders.*

Though a full proof is beyond the scope of this survey, we sketch the method for achieving the separation for a formula α in which there is a single occurrence of an S within the scope of a U . This case (and its mirror image) is the first and most important in a general inductive proof.

To begin with, using conjunctive and disjunctive normal forms and such easy equivalences as:

$$\begin{aligned} U(p \vee q, t) &\leftrightarrow U(p, t) \vee U(q, t), \\ U(p, q \wedge r) &\leftrightarrow U(p, q) \wedge U(p, r), \\ \neg S(q, r) &\leftrightarrow S(\neg r, \neg q) \vee P' \neg r, \end{aligned}$$

we can achieve a reduction to the case where α has one of the forms:

1. $U(p \wedge S(q, r), t)$
2. $U(p, q \wedge S(r, t))$

For (1), an equivalent which is a truth-functional compound of pure formulas is provided by :

$$1'. \quad [(S(q, r) \vee q) \wedge U(p, r \wedge t)] \vee U(q \wedge U(p, r \wedge t), t)$$

For (2) we have:

$$2'. \quad \{[(S(r, t) \wedge t) \vee r] \wedge [U(p, t) \vee U(\beta, t)]\} \vee \beta$$

where β is: $F' \neg t \wedge U(p, q \vee S(r, t))$. This, despite its complexity, is purely future. The observant reader should be able to see how completeness is needed for the equivalence of (2) and w' .

Unfortunately, U and S take us no further, for Kamp proves:

PROPOSITION 22. *The set $\{U, S\}$ is not temporally complete over $(\mathbb{Q}, <)$.*

Without entering into details, we note that one undefinable operator is $O(\varphi)$ where φ says:

$$\begin{aligned} \exists y(x < y \wedge \forall z(x < z \wedge z < y \rightarrow \\ (\forall w(x < w \wedge w < z \rightarrow P_1((w)) \vee \forall w(z < w \wedge w < y \rightarrow P_2(w)))))) \end{aligned}$$

Over complete orders $O(\varphi)(p, q)$ amounts to $U(G'q \wedge (p \vee q), p)$.

J. Stavi has found two new operators U', S' and proved:

THEOREM 23. *The set $\{U, S, U', S'\}$ is temporally complete over total orders.*

Gabbay has greatly simplified the proof: the idea is to try to prove the separation property over arbitrary total orders, and see what operators one needs. One quickly hits on the right U', S' . The combinatorial details cannot detain us here.

What about axiomatisability for U, S -tense logic? Some years ago Kamp announced (but never published) finite axiomatisability for various classes of total orders. Some are treated in [Burgess, 1982], where the system for dense orders takes a particularly simple form: we depart from standard format only to the extent of taking U, S as our primitives. As characteristic axioms, it suffices to take the following and their mirror images:

$$\begin{aligned} G(p \rightarrow q) &\rightarrow (U(p, r) \rightarrow U(q, r)) \wedge ((U(r, p) \rightarrow U(r, q)) \\ p \wedge U(q, r) &\rightarrow U(q \wedge S(p, r), r), \\ U(p, q) &\leftrightarrow U(p, q \wedge U(p, q)) \leftrightarrow U(q \wedge U(p, q), q), \\ U(, q) \wedge \neg U(p, r) &\rightarrow U(q \wedge \neg r, q), \\ U(p, q) \wedge U(r, s) &\rightarrow U(p \wedge r, q \wedge s \vee U(p \wedge s, q \wedge s) \vee U(q \wedge r, q \wedge s). \end{aligned}$$

A particularly important axiomatisability result is in [Gabbay *et al.*, 1980].

What about decidability? Rabin's theorem applies in most cases, the notable exceptions being complete orders, continuous orders, and $(\mathbb{R}, <)$. Here techniques of monadic second-order logic are useful. Decidability for the cases of complete and continuous orders is established in [Gurevich, 1977, Appendix]; and for $(\mathbb{R}, <)$ in [Burgess and Gurevich, 1985]. A fact (due to Gurevich) from the latter paper worth emphasising is that the U, S -tense logics of $(\mathbb{R}, <)$ and of arbitrary continuous orders are *not* the same.

5.2 Now, Then

We have seen that simple G, H -tense logic is inadequate to express certain temporal operators expressible in English. Indeed it turns out to be inadequate to express even the shortest item in the English temporal vocabulary, the word 'now'. Just what role this word plays is unclear—some incautious writers have even claimed it is semantically redundant—but [Kamp, 1971] gives a thorough analysis. Let us consider some examples:

0. The seismologist predicted that there would be an earthquake.
1. The seismologist predicted that there would be an earthquake *now*.
2. The seismologist predicted that there would already have been an earthquake *before now*.
3. The seismologist predicted that there would be an earthquake, but not till *after now*.

As Kamp says:

The function of the word ‘now’ in (1) is to make the clause to which it applies—i.e. ‘there would be an earthquake’—refer to the moment of utterance of (1) and not to the moment of moments (indicated by other temporal modifiers that occur in the sentence) to which the clause would refer (as it does in (0)) if the word ‘now’ were absent.

5.3 Formal Semantics

To formalise this observation, we introduce a new one-place connective J (for *jetzt*). We define a *pointed frame* to be a frame with a designated element. A *valuation* in a pointed frame (X, R, x_0) is just a valuation in $(X < R)$. We extend the definition of 0.4 above to G, H, J -formulas by adding the clause:

$$V(J\alpha) = X \text{ if } x_0 \in V(\alpha), \emptyset \text{ if } x_0 \notin V(\alpha)$$

is *valid* in (X, R, x_0) if $x_0 \in V(\alpha)$ for all valuations V .

An alternative approach is to define a *2-valuation* in a frame (X, R) to be a function assigning each p_i a subset of the Cartesian product X^2 . Parallel to 1.4 above we have the following inductive definition:

$$\begin{aligned} V(\neg\alpha) &= X^2 - V(\alpha), \\ V(\alpha \wedge \beta) &= V(\alpha) \cap V(\beta), \\ V(G\alpha) &= \{(x, y) : \forall x'(xRx' \rightarrow (x', y) \in V(\alpha))\}, \\ V(H\alpha), &\text{ similarly,} \\ V(J\alpha) &= \{(x, y) : (y, y) \in V(\alpha)\} \end{aligned}$$

α is *valid* in (X, R) if $\{(y, y) : y \in X\} \subseteq V(\alpha)$ for all 2-valuations V .

The two alternatives are related as follows: Given a 2-valuation V in the frame (X, R) , for each $y \in X$ consider the valuation V_y in the pointed frame $(X < R, y)$ given by $V_y(p_i) = \{x : (x, y) \in V(p_i)\}$. Then we always have $(y, y) \in V(\alpha)$ iff $y \in V_y(\alpha)$.

The second approach has the virtue of making it clear that though J is not a temporal operator in the sense of the preceding section, it is in a sense that can be made precise a *two-dimensional* tense operator. This suggests the project of investigating two- and multi-dimensional operators generally. Some such operators, for instance the ‘then’ of [Vlach, 1973], have a natural reading in English. Among other items in our bibliography, [Gabbay, 1976] and [Gabbay and Guenther, 1982] contain much information on this topic.

Using J we can express (0)–(3) as follows:

- 0'. P (seismologist says: F (earthquake occurs)),
- 1'. P (seismologist says: J (earthquake occurs)),

2'. P (seismologist says: JP (earthquake occurs)),

3'. P (seismologist says: JF (earthquake occurs)).

The observant reader will have noted that (0')–(3') are not really representable by G, H, J -formulas since they involve the notion of 'saying' or 'predicting'), a *propositional attitude*. Gabbay, too, gives many examples of uses of 'now' and related operators, and on inspection these, too, turn out to involve propositional attitudes. That this is no accident is shown by the following result of Kamp:

THEOREM 24 (Eliminability theorem). *For any G, H, J -formula α there is a G, H -formula α^* equivalent over all pointed frames.*

Proof. Call a formula *reduced* if it contains no occurrence of a J within the scope of a G or an H . Our first step is to find for each formula α an equivalent reduced formula α_R . This is done by induction on the complexity of α , only the cases $\alpha = G\beta$ or $\alpha = H\beta$ being nontrivial. In, for instance, the latter case, we use the fact that any truth-function can be put into disjunctive normal form, plus the following valid equivalence:

$$(R) \quad {}_H((Jp \wedge q) \wedge r) \leftrightarrow ((Jp \wedge H(q \vee r)) \vee (\neg Jp \wedge Hr))$$

Details are left to the reader. Our second step is to observe that if β is reduced, then it is equivalent to the result β^- of dropping all its occurrences of J . It thus suffices to set $\alpha^* = (\alpha_R)^-$. ■

The foregoing theorem says that in the presence of truth-functions and G and H , the operator J is, in a sense, redundant. By contrast, examples (0)–(3) suggest that in contexts with propositional attitudes, J is *not* redundant; the lack of a generally-accepted formalisation of the logic of propositional attitudes makes it impossible to turn this suggestion into a rigorous theorem. But in contexts with *quantifiers*, Kamp *does* prove rigorously that J is irredundant. Consider:

4. The Academy of Arts rejected an applicant who was to become a terrible dictator and start a great war.

5. The Academy of arts has rejected an applicant who is to become a terrible dictator and start a great war.

The following formalisations suggest themselves:

4'. $P(\exists x(R(x) \wedge FD(x)))$

5'. $P(\exists x(R(x) \wedge JFD(x)))$,

the difference between (4) and (5) lying precisely in the fact that the latter, unlike the former, definitely places the dictatorship and war in the hearer's future. What Kamp proves is that (5') cannot be expressed by a G, H -formula with quantifiers.

Returning to sentential tense logic, Theorem 24 obviously reduces the *decision* problem for G, H, J -tense logic to that for G, H -tense logic. As for *axiomatisability*, obviously we cannot adopt the standard format of G, H -tense logic, since the rule TG does not preserve validity for G, H, J -formulas. For instance:

$$(D0) \quad p \leftrightarrow Jp$$

is valid, but $G(p \leftrightarrow Jp)$ and $H(p \leftrightarrow Jp)$ are not. Kamp overcomes this difficulty, and shows how, in very general contexts, to obtain from a complete axiomatisation of a logic without J , a complete axiomatisation of the same logic with J . For the sentential G, H, J -tense logic of total orders, the axiomatisation takes a particularly simple form: take as sole rule MP. Let Lp abbreviate $Hp \wedge p \wedge Gp$. Take as axioms all substitution instances of tautologies, of (D0) above, and of $L\alpha$, where α may be any item on the lists (D1), (D2) below, or the mirror image of such an item:

$$(D1) \quad \begin{aligned} &G(p \rightarrow q) \rightarrow (Gp \rightarrow Gq) \\ &p \rightarrow Gp \\ &Gp \leftrightarrow GGp \\ &Lp \leftrightarrow GHp \end{aligned}$$

$$(D2) \quad \begin{aligned} &J\neg p \leftrightarrow \neg Jp \\ &J(p \wedge q) \leftrightarrow Jp \wedge Jq \\ &\neg L\neg Jp \leftrightarrow LJp \\ &Lp \rightarrow Jp. \end{aligned}$$

(In outline, the proof of completeness runs thus: using (D1) one deduces $Lp \rightarrow LLp$. It follows that the class of theses deducible without use of (D0) is closed under TG. Our work in Section 3.2 shows that we then get the complete G, H -tense logic of total orders. We then use (D2) to prove the equivalence (R) in the proof of Theorem 24 above. More generally, for any α , $\alpha \leftrightarrow \alpha_R$ is deducible without using (D0). Moreover, using D0, $\beta \leftrightarrow \beta^-$ is deducible for any reduced formula β . Thus in general $\alpha \leftrightarrow \alpha^*$ is a thesis, completing the proof.)

6 TIME PERIODS

The geometry of Space can be axiomatised taking unextended *points* as basic entities, but it can equally well be axiomatised by taking as basic certain regular open solid regions such as *spheres*. Likewise, the order of

Time can be described either (as in Section 1.1) in terms of *instants* in terms of *periods* of non zero duration. Recently it has become fashionable to try to redo tense logic, taking periods rather than instants as basic. Humberstone [1979] seems to be the first to have come out in print with such a proposal. This approach has become so popular that we must give at least a brief account of it; further discussion can be found in [van Benthem, 1991]. (See also Kuhn's discussion in the last chapter of this Volume of the *Handbook*.)

In part, the switch from instants to periods is motivated by a desire to model certain features of natural language. One of these is *aspect*, the verbal feature which indicates whether we are thinking of an occurrence as an *event* whose temporal stages (if any) do not concern us, or as a protracted *process*, forming, perhaps the backdrop for other occurrences. These two ways of looking at death (a popular, if morbid, example) are illustrated by:

When Queen Anne died, the Whigs brought in George.

While Queen Anne was dying, the Jacobites hatched treasonable plots.

Another feature of linguistic interest is the peculiar nature of *accomplishment* verbs, illustrated by:

1. The Amalgamated Conglomerate Building was built during the period March–August 1972.
- 1'. The ACB was built during the period April–July, 1972.
2. The ACB was being built (i.e. was under construction) during the period March–August, 1972.
- 2'. The ACB was under construction during the period April– July, 1972.

Note that (1) and (1') are inconsistent, whereas (2) implies (2')!

In part, the switch is motivated by a philosophical belief that periods are somehow more basic than instants. This motivation would be more convincing were 'periods' not assumed (as they are in too many recent works) to have sharply-defined (i.e. *instantaneous*) beginnings and ends. It may also be remarked that at the level of *experience* some occurrences do *appear* to be instantaneous (i.e. we don't *discern* stages in them). Thus 'bubbles when they burst' *seem* to do so 'all at once and nothing first'. While at the level of *reality*, some occurrences of the sort studied in quantum physics may well take place instantaneously, just as some elementary particles may well be pointlike. Thus the philosophical belief that every occurrence takes some time (period) to occur is not *obviously* true on any level.

Now for the mechanics of the switch: for any frame (X, R) we consider the set $I(X, R)$ of nonempty bounded open intervals of form $\{z : xRz \wedge zRy\}$.

Among the many relations on this set that could be defined in terms of R we single out two:

$$\begin{aligned} \text{Inclusion: } a \subseteq b & \quad \text{iff} \quad \forall x(x \in a \rightarrow x \in b), \\ \text{Order: } a \triangleleft b & \quad \text{iff} \quad \forall x \forall y(x \in a \wedge y \in b \rightarrow xRy). \end{aligned}$$

To any class $\mathcal{K}$ of frames we associate the class $\mathcal{K}'$ of those structures of form $(I(X, R), \subseteq, \triangleleft)$ with $(X, R) \in \mathcal{K}$, and the class $\mathcal{K}^+$ of those structures (Y, S, T) that are isomorphic to elements of $\mathcal{K}'$.

A first problem in switching from instants to periods as the basis for the logic of time is to find each important class $\mathcal{K}$ of frames a set of postulates whose models will be precisely the structures in $\mathcal{K}^+$. For the case of dense total orders without extrema, and for some other cases, suitable postulate sets are known, though none is very elegant. Of course this first problem is not yet a problem of *tense* logic; it belongs rather to applied first- and second-order logic.

To develop a period-based *tense* logic we define a *valuation* in a structure (Y, S, T) —where S, T are binary relations on Y —to be a function V assigning each p_i a subset of Y . Then from among all possible connectives that could be defined in terms of S and T , we single out the following:

$$\begin{aligned} V(\neg\alpha) &= Y - V(\alpha) \\ V(\alpha \wedge \beta) &= V(\alpha) \cap V(\beta) \\ V(\nabla\alpha) &= \{a : \forall b(bSa \rightarrow b \in V(\alpha))\} \\ V(\Delta\alpha) &= \{a : \forall b(aSb \rightarrow b \in V(\alpha))\} \\ V(F\alpha) &= \{a : \exists b(aTb \wedge b \in V(\alpha))\} \\ V(P\alpha) &= \{a : \exists b(bTa \wedge b \in V(\alpha))\}. \end{aligned}$$

The main *technical* problem now is, given a class $\mathbf{L}$ of structures (Y, S, T) —for instance, one of form $\mathbf{L} = \mathcal{K}^+$ for some class $\mathcal{K}$ of frames—to find a sound and complete axiomatisation for the tense logic of $\mathbf{L}$ based on the above connectives. Some results along these lines have been obtained, but none as definitive as those of instant-based tense logic reported in Section 3. Indeed, the choice of relations ($\subseteq$ and $\triangleleft$), and of admissible classes $\mathbf{L}$ (should we only consider classes of form $\mathcal{K}^+$?), and of connectives ($\neg, \wedge, \Delta, \nabla, F, P$), and of admissible valuations (should we impose restrictions, such as requiring $b \in V(p_i)$ whenever $a \in V(p_i)$ and $b \subseteq a$?) are all matters of controversy.

The main problem of *interpretation*—one to which advocates of period-based tense logic have perhaps not devoted sufficient attention—is how to make intuitive sense of the notion $a \in V(p)$ of a sentence p being true *with respect to* a time-period a . One proposal is to take this as meaning that p is true *throughout* a . Now given a valuation W in a frame (X, R) , we can define a valuation $I(W)$ in $I(X, R)$ by $I(W)(p_i) = \{a : a \subseteq W(p_i)\}$. When and *only* when V has the form $I(W)$ is ‘ p is true throughout a ’ a tenable reading of $a \in V(p)$. It is not, however, easy to characterise intrinsically

those V that admit a representation in the form $V = I(W)$. Note that even in this case, $a \in V(\neg p)$ does not express ‘ $(\neg p)$ is true throughout a ’ (but rather ‘ $\neg(p)$ is true throughout a ’). Nor does $a \in V(p \vee q)$ express ‘ $(p \vee q)$ is true throughout a ’.

Another proposal, originating in [Burgess, 1982] is to read $a \in V(p)$ as ‘ $+$ is *almost* always true during a ’. This reading is tenable when V has the form $J(W)$ for some valuation W in (X, R) , where $J(W)(p_i)$ is by definition $\{a : a - W(p_i) \text{ is nowhere dense in the order topology on } (XR)\}$. In this case, ‘ $(\neg p)$ is almost always true during a ’ is expressible by $a \in V(\nabla \neg p)$, and ‘ $(p \vee q)$ is almost always true during a ’ by $a \in V(\nabla \neg \nabla \neg(p \vee q))$. But the whole problem of interpretation for period-based tense logic deserves more careful thought.

There have been several proposals to redo tense logic on the basis of 3- or 4- of multi-valued truth-functional logic. It is tempting, of instance, to introduce a truth-value ‘unstatable’ to apply to, say, ‘Bertrand Russell is smiling’ in 1789. In connection with the switch from instants to periods, some have proposed introducing new truth-values ‘changing from true to false’ and ‘changing from false to true’ to apply to, say, ‘the rocket is at rest’ at take-off and landing times. Such proposals, along with proposals to combine, say, tense logic and intuitionistic logic, lie beyond the scope of this survey.

7 GLIMPSES AROUND

7.1 Metric Tense Logic

In *metric* tense logic we assume Time has the structure of an ordered Abelian group. We introduce variables $x, y, z, \dots$ ranging over group elements, and symbols $0, +, <$ for the group identity, addition, and order. We introduce operators $\mathcal{F}, \mathcal{P}$ joining terms for group elements with formulas. Here, for instance, $\mathcal{F}(x + y)(p \wedge q)$ means that it will be the case $(x + y)$ time-units hence that p and q . Metric tense logic is intended to reflect such ordinary-language *quantitative* expressions as ‘10 years from now’ or ‘tomorrow about this time’ or ‘in less than five minutes’. The *qualitative* F, P of nonmetric tense logic can be recovered by the definitions $Fp \leftrightarrow \exists x > 0 \mathcal{F}xp$, $Pp \leftrightarrow \exists x > 0 \mathcal{P}xp$. Actually, the ‘ago’ operator $\mathcal{P}$ is definable in terms of the ‘hence’ operator $\mathcal{F}$ since $\mathcal{P}xp$ is equivalent to $\mathcal{F} - xp$. It is not hard to write down axioms for metric tense logic whose completeness can be proved by a Henkin-style argument.

But decidability is lost: the decision problem for metric tense logic is easily seen to be equivalent to that for the set of all universal monadic (second- order) formulas true in all ordered Abelian groups. We will show that the decision problem for the validity of first-order formulas involving

a single two-place predicate $\in$ —which is well known to be unsolvable—is reducible to the latter: given a first-order $\in$ -formula φ , fix two one-place predicate variables U, V . Let φ_0 be the result of restricting all quantifiers in φ to U (i.e. $\forall x$ is replaced $\forall x(U(x) \rightarrow \dots)$ and $\exists x$ by $\exists x(U(x) \wedge \dots)$.) Let φ_1 be the result of replacing each atomic subformula $x \in y$ of φ_0 by $\exists z(V(z) \wedge V(z+x) \wedge V(z+x+y))$. Let φ_2 be the universal monadic formula $\forall U \forall V (\exists x U(x) \rightarrow \varphi_1)$. Clearly if φ is logically valid, then so is φ_2 and, in particular, the latter is true in all ordered Abelian groups. If φ is not logically valid, it has a countermodel consisting of the positive integers equipped with a binary relation E . Consider the product $\mathbb{Z} \times \mathbb{Z}$ where $\mathbb{Z}$ is the additive group of integers; addition in this group is defined by $(x, y) + (x', y') = (x+x', y+y')$; the group is orderable by $(x, y) < (x', y')$ iff $x < x'$ or $(x = x' \text{ and } y < y')$. Interpret U in this group as $\{(n, 0) : n > 0\}$; interpret V as the set consisting of the $(2^m 3^n, 0)$, $(2^m 3^n, m)$ and $(2^m 3^n, m+n)$ for those pairs (m, n) with mEn . This gives a countermodel to the truth of φ_2 in $\mathbb{Z} \times \mathbb{Z}$. Thus the desired reduction of decision problems has been effected.

Metric tense logic is, in a sense, a hybrid between the ‘regimentation’ and ‘autonomous tense logic’ approaches to the logic of time. Other hybrids of a different sort—not easy to describe briefly—are treated in an interesting paper of [Bull, 1978].

7.2 Time and Modality

As mentioned in the introduction, Prior attempted to apply tense logic to the exegesis of the writings of ancient and mediaeval philosophers and logicians (and for that matter of modern ones such as C. S. Peirce and J. Łukasiewicz) on future contingents. The relations between tense and mode or modality is properly the topic of Richmond H. Thomason’s chapter in this volume.

We can, however, briefly consider here the topic of so-called *Diodorean* and *Aristotelian* modal fragments of a tense logic L . The former is the set of modal formulas that become theses of L when $\Box p$ is defined as $p \wedge Gp$; the latter is the set of modal formulas that becomes theses of L when $\Box p$ is defined as $Hp \wedge p \wedge Gp$. Though these seem far-fetched definitions of ‘necessity’, the attempt to isolate the modal fragments of various tense logics undeniable was an important stimulus for the earlier development of our subject. Briefly the results obtained can be tabulated as follows. It will be seen that the modal fragments are usually well-known C. I. Lewis systems.

Class of frames	Tense logic	Diodorean fragment	Aristotelian fragment
All frames	$\mathbf{L}_0$	$\mathbf{T}(=\mathbf{M})$	$\mathbf{B}$
Partial orders	$\mathbf{L}_1$	$\mathbf{S4}$	$\mathbf{B}$
Lattices	$\mathbf{L}_0$	$\mathbf{S4.2}$	$\mathbf{B}$
Total orders	$\mathbf{L}_2, \mathbf{L}_5$	$\mathbf{S4.3}$	$\mathbf{S5}$
Dense orders			

The Diodorean fragment of the tense logic $\mathbf{L}_6$ of discrete orders has been determined by M. Dummett; the Aristotelian fragment of the tense logic of trees has been determined by G. Kessler. See also our comments below on R. Goldblatt's work.

7.3 *Relativistic Tense Logic*

The *cosmic* frame is the set of all point-events of space-time equipped with the relation of *causal accessibility*, which holds between u and v if a signal (material or electromagnetic) could be sent from u to v . The $(n+1)$ -dimensional *Minkowski* frame is the set of $(n+1)$ -tuples of real numbers equipped with the relation which holds between $(a_0, a_1, \dots, a_n)$ and $(b_0, b_1, \dots, b_n)$ iff:

$$\sum_{i=1}^n (b_i - a_i)^2 - (b_0 - a_0)^2 > 0 \text{ and } b_0 > a_0.$$

For present purposes, the content of the *special theory of relativity* is that the cosmic frame is isomorphic to the 4-dimensional Minkowski frame.

A little calculating shows that any Minkowski frame is a lattice without maximum or minimum, hence the tense logic of special relativity should at least include $\mathbf{L}_0$. Actually we will also want some axioms to express the density and continuity of a Minkowski frame. A surprising discovery of Goldblatt [1980] is that the *dimension* of a Minkowski frame influences its tense logic. Indeed, he shows that for each n there is a formula γ_{n+1} which is valid in the $(m+1)$ -dimensional Minkowski frame iff $m < n$. For example, writing Ep for $p \wedge Fp$, γ_2 is:

$$Ep \wedge Eq \wedge Er \wedge \neg E(p \wedge q) \wedge \neg E(p \wedge r) \wedge \neg E(q \wedge r) \rightarrow E((Ep \wedge Eq) \vee (Ep \wedge Er) \vee (Eq \wedge Er)).$$

On the other hand, he also shows that the dimension of a Minkowski frame does *not* influence the diodorean modal fragment of its tense logic: the Diodorean modal logic of special relativity is the same as that of arbitrary lattices, namely $\mathbf{S4.2}$. Combining Goldblatt's argument with the 'trousers world' construction in general relativity, should produce a proof that the Diodorean modal fragment of the latter is the same as that of arbitrary partial orders, namely $\mathbf{S4}$.

Despite recent advances, the tense logic of special relativity has not yet been completely worked out; that of general relativity is even less well understood. Burgess [1979] contains a few additional philosophical remarks.

7.4 *Thermodynamic Time*

One of the oldest metaphysical concepts (found in Hindu theology and pre-Socratic philosophy, and in modern psychological dress in Nietzsche and celestial mechanical dress in Poincaré) is that everything that has ever happened is destined to be repeated over and over again. This leads to a degenerate tense logic containing the principles $Gp \rightarrow Hp$ and $Gp \rightarrow p$ among others.

An antithetical view is that traditionally associated with the Second Law of Thermodynamics, according to which irreversible change are taking place that will eventually drive the Universe to a state of ‘heat-death’, after which no further change on a macroscopically observable level will take place. The tense logic of this view, which raises several interesting technical points, has been investigated by S. K. Thomason [1972]. The first thing to note is that the principle:

$$(A10) \quad GFp \rightarrow FGp$$

is acceptable for p expressing propositions about macroscopically observable states of affairs provided these do not contain hidden time references; e.g. p could be ‘there is now no life on Earth’, but not ‘particle κ currently has a momentum of precisely k gram-meters/second’ or ‘it is now an even number of days since the Heat Death occurred’. For the antecedent of (A20) says that arbitrarily far in the future there will be times when p is the case. But for the p that concern us, the truth-value of p is never supposed to change after the Heat Death. So in that case, there will come a time after which p is always going to be true, in accordance with the consequent of (A10).

The question now arises, how can we *formalise* the restriction of p to a special class of sentences? In general, propositions are represented in the formal semantics of tense logic by subsets of X in a frame (X, R) . A restricted class of propositions could thus be represented by a distinguished family $\mathcal{B}$ of subsets of X . This motivates the following definition: an *augmented* frame is a triple $(X, R, \mathcal{B})$ where (X, R) is a frame, $\mathcal{B}$ a subset of the lower set $\mathcal{B}(X)$ of X closed under complementation, finite intersection, and the operations:

$$\begin{aligned} gA &= \{x \in X : \forall y \in X (xRy \rightarrow y \in A)\} \\ hA &= \{x \in X : \forall y \in X (yRx \rightarrow y \in A)\}. \end{aligned}$$

A *valuation* in $(X, R, \mathcal{B})$ is a function V assigning each variable p_i an element of $\mathcal{B}$. The closure conditions on $\mathcal{B}$ guarantee that we will then have $V(\alpha) \in \mathcal{B}$

for *all* formulas α . It is now clear how to define validity. Note that if $\mathcal{B} = \mathcal{P}(X)$, then the validity in $(X, R, \mathcal{B})$ reduces to validity in (X, R) ; otherwise more formulas may be valid in the former than the latter.

It turns out that the extension $\mathbf{L}_{10}$ of $\mathbf{L_Q}$ obtained by adding (A10) is (sound and) complete for the class of augmented frames $(X, R, \mathcal{B})$ in which (X, R) is a dense total order without maximum or minimum and:

$$\forall B \in \mathcal{B} \exists x (\forall y (xRy \rightarrow y \in B) \vee \forall y (xRy \rightarrow y \notin B)).$$

We have given complete axiomatisations for many intuitively important classes of frames. We have not yet broached the questions: when does the tense logic of a given class of frames admit a complete axiomatisation? When does a given axiomatic system of tense logic correspond to some class of frames in the sense of being complete for that class? For information on these large questions, and for bibliographical references, we refer the reader to Johan van Benthem's chapter in Volume 3 of this edition of the *Handbook* on so-called 'Correspondence Theory'. Suffice it to say here that positive general theorems are few, counterexamples many. The thermodynamic tense logic $\mathbf{L}_{10}$ exemplifies one sort of pathology. Though it is not inconsistent, there is *no* (unaugmented) frame in which all its theses are valid!

7.5 Quantified Tense Logic

The interaction of temporal operators with universal and existential quantifiers raises many difficult issues, both philosophical (over identity through changes, continuity, motion and change, reference to what no longer exists or does not exist, essence, and many, many more) and technical (over undecidability, nonaxiomatisability, undefinability or multi-dimensional operators, and so forth) that it is pointless to attempt even a survey of the subject in a paragraph or two. We therefore refer the reader to Nino Cocchiarella's chapter in this volume and James W. Garson's chapter in Volume 3 of this edition of the *Handbook*, both on this subject.

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ADVANCED TENSE LOGIC

1 INTRODUCTION

In this chapter we consider the tense (or temporal) logic with until and since connectives over general linear time. We will call this logic *US/LT*. This logic is an extension of Prior's original temporal logic of *F* and *P* over linear time [Prior, 1957], via the introduction of the more expressive connectives of Kamp's *U* for “until” and *S* for “since” [Kamp, 1968b]. *U* closely mimics the natural language construct “until” with $U(A, B)$ holding when *A* is constantly true from now up until a future time at which *B* holds. *S* is similar with respect to the past. We will see that *U* and *S* do indeed extend the expressiveness of the temporal language.

In the chapter we will also be looking at other related temporal logics. The logics differ from each other in two respects. Logics may differ in the kinds of structures which they are used to describe. Structures vary in terms of their underlying model of time (or frame): this can be like the natural numbers, or like the rationals or like the reals or some other linear order or some non-linear branching or multi-dimensional shape. Logics are defined with respect to a class of structures. Considering a logic defined by the class of all linear structures is a good base from which to begin our exploration. Temporal logics also vary in their language. For various purposes, until and since may be not expressive enough. For example, if we want to be able to reason about alternative avenues of development then we may want to allow branches in the flow of time and, in order to represent directly the fact of alternative possibilities, we may need to add appropriate branching connectives. Equally, until and since may be too strong: for simple reasoning about the forward development of a mechanical system, using since may not only be unnecessary, but may require additional axioms and complexity of a decision procedure.

In this chapter we will not be looking at temporal logics based on branching. See the handbook chapter by Thomason for these matters. We will also avoid consideration of temporal logics incorporating quantification. Instead, see the handbook chapter by Garson for a discussion of predicate temporal and modal logics and see the reference [Gabbay *et al.*, 1994] for a discussion of temporal logics incorporating quantification over propositional atoms.

So we will begin with a tour of the many interesting results concerning *US/LT* including axiom systems, related logics, decidability and complexity. In section 3 we sketch a proof of the expressive completeness of the logic.

Then, in section 4 we investigate combinations of logics with a temporal element. In section 5, we develop the proof theory for temporal logic within the framework of labelled deductive systems. In section 6, we show how temporal reasoning can be handled within logic programming. In section 7, we survey the much studied temporal logic of the natural numbers and consider the powerful automata technique for reasoning about it. Finally, in section 8, we consider the possibility of treating temporal logic in an imperative way.

2 U, S LOGIC OVER GENERAL LINEAR TIME

Here we have a close look at the US logic over arbitrary linear orders.

2.1 *The logic*

Frames for our logic are linear. Thus we have a non-empty set T and a binary relation $< \subseteq T \times T$ which is:

1. *irreflexive*, i.e. $\forall t \in T$, we do not have $t < t$;
2. *total*, i.e. $\forall s, t \in T$, either $s < t$, $s = t$ or $t < s$;
3. *transitive*, i.e. $\forall s, t, u \in T$, if $s < t$ and $t < u$ then $s < u$.

The underlying model of time for a temporal logic is captured by the frame $(T, <)$.

Any use of a temporal logic will involve something happening over time. The simplest method of trying to capture this formally is to use a propositional temporal logic. So we fix a countable set $\mathcal{L}$ of atoms. The truth of a particular atom will vary in time. For example, points of time (i.e. $t \in T$) may correspond to days and the truth of the atom r on a particular day may correspond to the event of rain on that day.

A structure is a particular history of the truth of all the atoms over the full extent of time. Structures $(T, <, h)$ are linear so we have a linear frame $(T, <)$ and we have a valuation h for the atoms, i.e. for each atom $p \in \mathcal{L}$, $h(p) \subseteq T$. The set $h(p)$ is the set of all time points at which p is true.

The language $L(U, S)$ is generated by the 2-place connectives U, S along with classical $\neg$ and $\wedge$. That is, we define the set of formulas recursively to contain the atoms and $\top$ (i.e. truth) and for formulas A and B we include $\neg A$, $A \wedge B$, $U(A, B)$ and $S(A, B)$. We read $U(A, B)$ as “until A, B ” corresponding to B being true until A is. Similarly S is read as “since”.

Formulas are evaluated at points in structures. We write $\mathcal{T}, x \models A$ when A is true at the point $x \in T$. This is defined recursively as follows. Suppose that we have defined the truth of formulas A and B at all points of $\mathcal{T}$. Then for all points x :

$\mathcal{T}, x \models p$	iff	$x \in h(p)$, for p atomic;
$\mathcal{T}, x \models \top$;		
$\mathcal{T}, x \models \neg A$	iff	$\mathcal{T}, x \not\models A$;
$\mathcal{T}, x \models A \wedge B$	iff	both $\mathcal{T}, x \models A$ and $\mathcal{T}, x \models B$;
$\mathcal{T}, x \models U(A, B)$	iff	there is a point $y > x$ in T such that $\mathcal{T}, y \models A$ and for all $z \in T$ such that $x < z < y$ we have $\mathcal{T}, z \models B$;
$\mathcal{T}, x \models S(A, B)$	iff	there is a point $y < x$ in T such that $\mathcal{T}, y \models A$ and for all $z \in T$ such that $y < z < x$ we have $\mathcal{T}, z \models B$;

Often definitions and results involving S can be given by simply exchanging U and S and swapping $<$ and $>$. In that situation we just mention that a *mirror image* case exists and do not go into details.

There are many abbreviations that are commonly used in the language. As well as the classical $\perp$ (i.e. $\neg\top$ for falsity), $\vee$, $\rightarrow$ and $\leftrightarrow$, we have the following temporal abbreviations:

FA	$= U(A, \top)$	A will happen (sometime);
GA	$= \neg F\neg A$	A will always hold;
PA	$= S(A, \top)$	A was true (sometime);
HA	$= \neg P\neg A$	A was always true;
$K^+(A)$	$= \neg U(\top, \neg A)$	A will be true arbitrarily soon;
$K^-(A)$	$= \neg S(\top, \neg A)$	A was true arbitrarily recently.

Notice that Prior's original connectives F and P appear as abbreviations in this logic. The reader should check that their original semantics (see [Burgess, 2001]) are not compromised.

A formula ϕ is *satisfiable* if it has a model: i.e. there is a structure $\mathcal{T} = (T, <, h)$ and $x \in T$ such that $\mathcal{T}, x \models \phi$. A formula is *valid* iff it is true at all points of all structures. We write $\models A$ iff A is a validity. Of course, a formula is valid iff its negation is not satisfiable.

We can also define (semantic) consequence in the logic. Suppose that Γ is a set of formulas and A a formula. We say that A is a consequence of Γ and write $\Gamma \models A$ iff whenever we have $\mathcal{T}, t \models C$ for all $C \in \Gamma$, for some point t from some structure $\mathcal{T}$, then we also have $\mathcal{T}, t \models A$.

First-Order Monadic Logic of Order

For many purposes such as assessing the expressiveness of temporal languages or establishing their decidability, it is useful to be able to move from the internal tensed view of the world to an external untensed view. In doing

so we can also make use of logics with more familiar syntax. In the case of our linear temporal logics we find it convenient to move to the first-order monadic logic of linear order which is a sub-logic of the full second-order monadic logic of linear order.

The language of the full second-order monadic logic of linear order has formulas built from $<$, $=$, quantification over individual variable symbols and quantification over monadic (i.e. 1-ary) predicate symbols. To be more formal, suppose that $X = \{x_0, x_1, \dots\}$ is our set of individual variable symbols and $Q = \{P_0, P_1, \dots\}$ is our set of monadic predicates. The formulas of the language are $x_i < x_j$, $x_i = x_j$, $P_i(x_j)$, $\neg\alpha$, $\alpha \wedge \beta$, $\exists x_i \alpha$, and $\exists P_j \alpha$ for any $i, j < \omega$ and any formula α . We use the usual abbreviations $x_i > x_j$, $x_i \leq x_j$, $x_i < x_j < x_k$, $\forall x_i \alpha$ and $\forall P_i \alpha$ etc.

As usual we define the concept of a free individual variable symbol in a formula. We similarly define the set of free monadic variables of a formula. Write $\phi(x_1, \dots, x_m, P_1, \dots, P_n)$ to indicate that all the free variables (of both sorts) in the formula ϕ are contained in the lists $x_1, \dots, x_m$ and $P_1, \dots, P_n$.

The language is used to describe linear orders. Suppose that $(T, <)$ is a linear order. As individual variable symbols we will often use t, s, r, u etc, instead of $x_1, x_2, \dots$.

An individual variable assignment V is a mapping from X into T . A predicate variable assignment W is a mapping from Q into $\wp(T)$ (the set of subsets of T). For an individual variable assignment V , an individual variable symbol $x \in X$ and an element $t \in T$, we define the individual variable assignment $V[x \mapsto t]$ by:

$$V[x \mapsto t](y) = \begin{cases} V(y) & y \neq x \\ t & y = x. \end{cases}$$

Similarly for predicate variable assignments and subsets of T .

For a formula ϕ , variable assignments V (individual) and W (predicate), we define whether (or not resp.) ϕ under V and W is true in $(T, <)$, written $(T, <), V, W \models \phi$ by induction on the quantifier depths of ϕ .

Given some ϕ , suppose that for all its subformulas ψ , for all variable assignments V and W , we have defined whether or not $(T, <), V, W \models \psi$. For variable assignments V and W we define:

$$\begin{aligned}
(T, <), V, W \models x_i < x_j & \text{ iff } V(x_i) < V(x_j); \\
(T, <), V, W \models x_i = x_j & \text{ iff } V(x_i) = V(x_j); \\
(T, <), V, W \models P_i(x_j) & \text{ iff } V(x_j) \in W(P_i); \\
(T, <), V, W \models \neg \alpha & \text{ iff } (T, <), V, W \not\models \alpha; \\
(T, <), V, W \models \alpha \wedge \beta & \text{ iff } (T, <), V, W \models \alpha \text{ and } (T, <), V, W \models \beta; \\
(T, <), V, W \models \exists x_i \alpha & \text{ iff } \text{there exists some } t \in T \text{ such that} \\
& (T, <), V[x_i \mapsto t], W \models \alpha. \\
(T, <), V, W \models \exists P_j \alpha & \text{ iff } \text{there is some } S \subseteq T \text{ such that} \\
& (T, <), V, W[P_j \mapsto S] \models \alpha.
\end{aligned}$$

This is standard second-order semantics. Note that it is easy to show that the truth of a formula does not depend on the assignment to variables which do not appear free in the formula.

Mostly we will be interested in fragments of the full second-order monadic logic. In particular, we refer to the first-order monadic logic of linear order which contains just those formulas with no quantification of predicate variables. We will also mention the universal second-order monadic logic of linear order which contains just those formulas which consist of a first-order monadic formula nested under zero or more universal quantifications of predicate variables.

The important correspondence for us is that between temporal logics such as US/LT and the first-order monadic logic. Most of the temporal logics which we will consider allow a certain equivalence between their formulas and first-order monadic formulas. To define this we need to use a fixed one-to-one correspondence between the propositional atoms of the temporal language and monadic predicate variables. Let us suppose that $p_i \in \mathcal{L}$ corresponds to $P_i \in \mathcal{Q}$.

The translation will propagate upwards through the full temporal language provided that each of the connectives have a first-order translation. In particular we require for any n -ary temporal connective C some first-order monadic formula $\phi_C(t, P_1, \dots, P_n)$ which corresponds to $C(p_1, \dots, p_n)$. We say that ϕ_C is the (*first-order*) *table* of C iff for every linear order $(T, <)$, for all $h : \mathcal{L} \rightarrow \wp(T)$, for all $t_0 \in T$, for all variable assignments V and W ,

$$\begin{aligned}
(T, <, h), t_0 & \models C(p_1, \dots, p_n) \\
& \text{ iff } (T, <), V[t \mapsto t_0], W[P_1 \mapsto h(p_1), \dots, P_n \mapsto h(p_n)] \models \phi_C.
\end{aligned}$$

U and S have first-order tables as follows: the table of U is $\phi_U = \exists s((t < s) \wedge P_1(s) \wedge \forall r((t < r \wedge r < s) \rightarrow P_2(r)))$.

The table of S is the mirror image.

If we have a temporal logic with first-order tables for its connectives then it is straightforward to define a meaning-preserving translation (to the first-order monadic language) of all formulas in the language. The translation is

as follows:

$$\begin{aligned}
* p_i &= P_i(t) \\
* \top &= (t = t) \\
* \neg A &= \neg * A \\
* (A \wedge B) &= * A \wedge * B \\
* U(A, B) &= \phi_U[*A/P_1][*B/P_2] \\
* S(A, B) &= \phi_S[*A/P_1][*B/P_2]
\end{aligned}$$

Here, if α is a first-order monadic formula with one free individual variable t , we use the notation $\phi[\alpha/P]$ to mean the result of replacing every (free) occurrence of the predicate variable symbol P in the first-order monadic formula ϕ by the monadic formula α : i.e. $P(x_i)$ gets replaced by $\alpha[x_i/t]$. This is a little complex as we must take care to avoid clashes of variable symbols. We will not go into details here.

For any temporal formula A , $*A$ is a first-order monadic formula with t being the only free individual variable symbol. If atom p_i appears in A then $*A$ will also have a free predicate symbol P_i . There are no other predicate symbols in $*A$.

We then have:

LEMMA 1. *For each linear order $(T, <)$, for any $t_0 \in T$, for any temporal A , if A uses atoms from $p_1, \dots, p_n$ then for all variable assignments V and W ,*

$$\begin{aligned}
(T, <, h), t_0 \models A \text{ iff } (T, <), V[t \mapsto t_0], W[P_1 \mapsto h(p_1)] \dots \\
[P_n \mapsto h(p_n)] \models *A.
\end{aligned}$$

Proof. By induction on the construction of A . ■

Below, when we discuss expressive power in section 3, we will consider whether there is a reverse translation from the first-order monadic language to the temporal language.

Uses of US/LT

In the previous chapter [Burgess, 2001], we saw a brief survey of the motivations for developing a tense logic. There are particular reasons for concentrating on the logic *US/LT*. It is a basis for reasoning about events and states in general linear time or particularly dense or continuous time.

For example, it is certainly the case that most use of tense and aspect in natural language occur in the context of an assumed dense linear flow of time. Any kind of reasoning about the same sorts of situations, as in many branches of AI or cognitive science, also requires a formalism based on dense, continuous or general linear time.

In computer science the most widely used temporal logic (PTL which we will meet later) is based on a discrete natural numbers model of time. However, there has been much recent work on developing logics based on more general models for many applications. Particular applications include refinement, analogue devices, open systems, and asynchronous distributed or concurrent systems.

Refinement here concerns the process of making something more concrete or more specific. This can include making a specification for a system (or machine, or software system) less ambiguous, or more detailed, or less nondeterministic. This can also include making an algorithm, or program, or implementation, or design, more detailed, or more low level. There are several different ways of producing a refinement but one involves breaking up one step of a process into several smaller steps which accomplish the same overall effect. If a formal description of a less refined process assumes discrete steps of time then it is easy to see that it may be hard to relate it to a description of a more refined process. Extra points of time may have to be introduced in between the assumed ones. Using a dense model of time from the outset will be seen to avoid this problem. Comparing the formal descriptions of more and less refined processes is essential for checking the correctness of any refinement procedure.

It can also be seen that a general linear model of time will be useful in describing analogue devices (which might vary in state in a continuous way), open systems (which might be affected by an unbounded number of different environmental events happening at all sorts of times) and asynchronous and distributed systems (which may have processes going through changes in state at all sorts of times).

In general the logics are used to describe or specify, to verify, to model-check, to synthesize, or to execute. A useful description of these activities can be found in [Emerson, 1990]. Specification is the task of giving a complete, precise and unambiguous description of the behaviour expected of a system (or device). Verification is the task of checking, or proving, that a system does conform to a specification and this includes the more specific task of model-checking, or determining whether a given system conforms to a particular property. Synthesis is the act of more or less automatically producing a correct system from a specification (so this avoids the need for verification). Finally execution is the process of directly implementing a specification, treating the specification language as an implementation (or programming) language.

In many of these applications it is crucial to determine whether a formula is a consequence of a set of formulas. For example, we may have a large and detailed set of formulas exactly describing the behaviour of the system and we have a small and very interesting formula describing a crucial desired property of the system (e.g., “it will fly”) and we want to determine whether the latter follows from the former. We turn to this question now.

2.2 An Axiomatization for the Logic

We have seen the importance of the consequence relation in applications of temporal logic. Because of this there are good reasons to consider syntactical ways of determining this relation between sets of formulas and a single formula. One of the most widely used and widely investigated such approaches is via a Hilbert system. Here we look in detail at a Hilbert style axiom system for our US/LT logic.

The system, which we will call $Ax(U, S)$, was first presented in [Burgess, 1982] but was later simplified slightly in [Xu, 1988]. It has what are the usual inference rules for a temporal logic: i.e. modus ponens and two generalizations, temporal generalization towards the future and temporal generalization towards the past:

$$\frac{A, A \rightarrow B}{B} \quad \frac{A}{GA} \quad \frac{A}{HA}$$

Each rule has a list of formulas (or just one formula) as antecedents shown above the horizontal line and a single formula, the consequent below the line. An *instance* of a rule is got by choosing any particular $L(U, S)$ formulas for the A and B . We describe the role of a rule below.

The axioms of $Ax(U, S)$ are all substitution instances of truth-functional tautologies and the following temporal schemas:

- (1) $G(A \rightarrow B) \rightarrow (U(A, D) \rightarrow U(B, D)),$
- (2) $G(A \rightarrow B) \rightarrow (U(D, A) \rightarrow U(D, B)),$
- (3) $A \wedge U(B, D) \rightarrow U(B \wedge S(A, D), D),$
- (4) $U(A, B) \rightarrow U(A, B \wedge U(A, B)),$
- (5) $U(B \wedge U(A, B), B) \rightarrow U(A, B),$
- (6) $U(A, B) \wedge U(D, E) \rightarrow$
 $(U(A \wedge D, B \wedge E) \vee U(A \wedge E, B \wedge E) \vee U(B \wedge D, B \wedge E)).$

along with the mirror images of (1) to (6). Notice that (1) and (2) are closely related to the usual axioms for the modal logic K , (3) relates the mirror image connectives, (4) and (5) have something to do with transitivity and (6) captures an aspect of linearity.

We say that a formula B follows from a list $A_1, \dots, A_n$ of formulas by one of the rules of inference iff there is an instance of the rule with $A_1, \dots, A_n$ as the antecedents and B as the consequent. A deduction in $Ax(U, S)$ is a finite sequence of formulas with each being either an instance of one of the axioms or following from a list of previous formulas in the sequence by one of the rules of inference. Any formula which appears as the last element in a derivation is called a *thesis* of the system. If A is a thesis then we write $\vdash A$.

We say that a formula D is a (*syntactic*) *consequence* of a set Γ of formulas iff there are $C_1, \dots, C_n \in \Gamma$ such that $\vdash (\bigwedge_{i=1}^n C_i) \rightarrow D$. In that case we write $\Gamma \vdash D$. Note that in some other logics it is possible to use a Hilbert system to define consequence via the idea of a deduction from hypotheses. In general in temporal logic, we do not do so because of problems with the generalization rules.

Note that alternative presentations of such a system may use what is known as the substitution rule:

$$\frac{A}{A[B/q]}$$

We have a whole collection of instances of the substitution rule: one for each (formula, atom , formula) triple. If B is a formula and q is an atom then we define the substitution $A[B/q]$ of q by B in a formula A by induction on the construction of A . We simply have $q[B/q] = B$ and $p[B/q] = p$ for any atom p other than q . The induction then respects every other logical operator, e.g., $(A_1 \wedge A_2)[B/q] = (A_1[B/q]) \wedge (A_2[B/q])$. If we include the substitution rule in an axiom system then axioms can be given in terms of particular atoms. For example, we could have an axiom $G(p \rightarrow q) \rightarrow (U(p, r) \rightarrow U(q, r))$.

Soundness of The System

We will now consider the relation between syntactic consequence and semantic consequence.

We say that an axiom system is *sound* (with respect to a semantically defined logic) iff any syntactic consequence (of the axiom system) is also a semantic consequence. This can readily be seen to be equivalent to the property of every thesis being valid.

We can show that:

LEMMA 2. *The system $Ax(U, S)$ is sound for US/LT .*

Proof. Via a simple induction it is enough to show that every axiom is valid and that each rule of inference preserves validity, i.e. that the consequent is valid if all the antecedents are.

The axioms are straightforward to check individually using obvious semantic arguments.

The inference rules are equally straightforward to check. For example, let us look at temporal generalization towards the future. Suppose that A is valid. We are required to show that GA is valid. So suppose that $(T, <, g)$ is a linear structure and $t \in T$. Consider any $s > t$. By the validity of A we have $(T, <, g), s \models A$. This establishes that $(T, <, g), t \models GA$ as required. ■

Completeness of the System

An axiom system is said to be (sound and) *complete* for a logic iff syntactic consequence exactly captures semantic consequence.

In fact, this is sometimes called *strong completeness* because there is a weaker notion of completeness which is still useful. We say that an axiom system is *weakly complete* for a logic iff it is sound and every validity is a thesis. Clearly weak completeness follows strong completeness. As we will see, there are temporal logics for which we can only obtain weakly complete axiom systems.

However,

THEOREM 3. *$Ax(U, S)$ is strongly complete for US/LT .*

Proof. We sketch the proof from [Burgess, 1982].

We need some common definitions in order to proceed. We say that a set of formulas is *consistent* (in the context of an axiom system) iff it is not the case that $\Gamma \vdash \perp$. If Γ is maximal in being consistent, i.e. the addition to Γ of any other formula in the language would result in inconsistency, then we say that Γ is a maximal consistent set (MCS). A useful result, due to Lindenbaum (see [Burgess, 2001]), gives us,

LEMMA 4. *If Δ is a consistent set then there is an MCS $\Gamma \supseteq \Delta$.*

There are many useful properties of MCSs, e.g.,

LEMMA 5. *If Γ is an MCS then $A \in \Gamma$ iff $\neg A \notin \Gamma$.*

In order to show the completeness of the axiom system we need only show that each MCS is satisfiable. To see this suppose that $\Gamma \models A$. Thus $\Gamma \cup \{\neg A\}$ is unsatisfiable. It can not be consistent as then by lemma 4 it could be extended to an MCS which we know would be satisfiable. Thus we must be able to derive $\perp$ from $\Gamma \cup \{\neg A\}$, say $\bigwedge_{i=1}^n C_i \wedge \neg A \vdash \perp$. It is a simple matter to show that then we have $\vdash \bigwedge_{i=1}^n C_i \rightarrow A$, i.e. that A is a syntactic consequence of Γ .

So suppose that Γ_0 is an MCS: we want to show that it is satisfiable. We use the rationals as a base board on which we successively place whole maximal consistent sets of formulas as points which will eventually make up a flow of time. So, at each stage, we will have a subset $T \subseteq \mathbb{Q}$ and an MCS $\Gamma(t)$ for each $t \in T$.

Starting with our given maximal consistent set placed at zero, say, we look for counter-examples to either of the following rules:

1. if $U(A, B) \in \Gamma(t)$ then there should be some $\Gamma(s)$ placed at $s > t$ with $A \in \Gamma(s)$ and so that theories placed in between t and s all contain B and
2. if $\neg U(A, B) \in \Gamma(t)$ and $A \in \Gamma(s)$ placed at $s > t$ then there should be $\Gamma(r)$ placed somewhere in between with $\neg B \in \Gamma(r)$.

By carefully choosing a single maximal consistent set to right the counter-example and satisfy some other stringent conditions kept holding throughout the construction, we can ensure that the particular tuple $(t, U(A, B))$ or $(t, s, \neg U(A, B))$ never again forms a counter-example. In order to do so we need to record, for each interval between adjacent sets, which formulas must belong to any set subsequently placed in that interval. Because there are only countable numbers of points and formulas involved, in the limit we can effect that we end up with a counter-example-free arrangements of sets. This is so nice that if we define a valuation h on the final $T \subseteq \mathbb{Q}$ via $t \in h(p)$ iff $p \in \Gamma(t)$, then for all $t \in T$, for all $A \in L(U, S)$,

$$(T, <, h), t \models A \text{ iff } A \in \Gamma(t).$$

This is thus our model. ■

The IRR rule

Here we will examine a powerful alternative approach to developing Hilbert systems for temporal logics like US/LT . It is based on the use of rules such as the IRR (or irreflexivity) rule of [Gabbay, 1981]. Recall that a binary relation $<$ on a set T is irreflexive if we do not have $t < t$ for any $t \in T$. The IRR rule allows

$$\frac{q \wedge H(\neg q) \rightarrow A}{A} \quad \text{provided that the atom } q \text{ does not appear in the formula } A.$$

A short proof (see for example [Gabbay and Hodkinson, 1990], Proposition 2.2.1) establishes that,

LEMMA 6. *if $\mathcal{I}$ is a class of (irreflexive) linear orders, then IRR is a valid rule in the class of all structures whose underlying flow of time comes from $\mathcal{I}$.*

Proof. Suppose that $q \wedge H(\neg q) \rightarrow A$ is valid. Consider any structure $(T, <, h)$ with $(T, <)$ from $\mathcal{I}$ and any $t \in T$. Let g be a valuation of the atoms on T which is like h but differs only in that $g(q) = \{t\}$, i.e. $g = h[q \mapsto \{t\}]$. By the assumed validity, $(T, <, g) \models q \wedge H(\neg q) \rightarrow A$. But notice that also $(T, <, g) \models q \wedge H(\neg q)$. Thus $(T, <, g) \models A$. Clearly, since q does not appear in A , $(T, <, h) \models A$. Thus A is valid in the logic. ■

The original motivation for the use of this rule concerned the impossibility of writing an axiom to enforce irreflexivity of flows (see for example [van Benthem, 1991]). The usual technique in a completeness proof is to construct some model of a consistent formula and then turn it into an irreflexive model. IRR allows immediate construction of an irreflexive model. This is because it is always consistent to posit the truth of $q \wedge H(\neg q)$ (for some ‘new’ atom q) at any point as we do the construction.

The benefits of this rule for doing a completeness proof are enormous. Much use of it is made in [Gabbay *et al.*, 1994]. Venema [Venema, 1993] gives a long list of results proved using IRR or similar rules: examples include branching time logics [Zanardo, 1991] and two-dimensional modal logics [Kuhn, 1989]. In fact, the major benefit of IRR is a side-effect of its purpose. Not only can we construct a model which is irreflexive but we can construct a model in which each point has a unique name (as the first point where a certain atom holds).

As an example, consider our logic US/LT . We can, in fact mostly just use the standard axiomatization for F and P over the class of all linear orders. This is because if you have a unique name of the form $r \wedge H(\neg r)$ for each point then their axiom

$$(UU) \quad r \wedge H(\neg r) \rightarrow [U(p, q) \leftrightarrow F(p \wedge H[Pr \rightarrow q])]$$

and its mirror image (SS) essentially define U and S in terms of F and P .

So here is an axiom system for US/LT similar to those seen in [Gabbay and Hodkinson, 1990] and [Gabbay *et al.*, 1994]. Call it Z . The rules are modus ponens, two generalizations, substitution:

$$\frac{A, A \rightarrow B}{B} \quad \frac{A}{GA} \quad \frac{A}{HA} \quad \frac{A}{A[B/q]}$$

and the IRR:

$$\frac{q \wedge H(\neg q) \rightarrow A}{A} \quad \text{provided that the atom } q \text{ does not appear in the formula } A.$$

The axioms are:

1. all truth functional tautologies,
2. $G(p \rightarrow q) \rightarrow (Gp \rightarrow Gq)$,
3. $Gp \rightarrow GGp$,
4. $G(p \wedge Gp \rightarrow q) \vee G(q \wedge Gq \rightarrow p)$,
5. $r \wedge H\neg r \rightarrow (U(p, q) \leftrightarrow F(p \wedge H(Pr \rightarrow q)))$,

And mirror images of the above.

THEOREM 7. *The axiom system is sound and (weakly) complete for US/LT .*

Soundness is the usual induction on the lengths of proofs.

For completeness we have to do quite a bit of extra work. However this extra work is quite general and can form the basis of many and varied completeness proofs. The general idea is along the lines of the usual Henkin-style completeness proof for Prior's logic over linear time (see [Burgess, 2001]) but there is no need of bulldozing of clusters. It is as follows.

Say that a set is Z -consistent iff there is no conjunction A of its formulas such that Z derives $A \rightarrow \perp$. We are interested in maximal consistent sets of a particular sort which we call IRR theories. They contain a ‘name’ of the form $\neg q \wedge Hq$ and also similarly name any other time referred to by some zig-zagging sequence of F s and P s.

DEFINITION 8. A theory Δ is said to be an IRR-theory if (a) and (b) hold:

- (a) For some q , $\neg q \wedge Hq \in \Delta$.
- (b) Whenever $X = \Diamond_1(A_1 \wedge \Diamond_2(A_2 \wedge \dots \wedge \Diamond_n A_n)) \dots \in \Delta$,
then for some new atom q ,
 $X(q) = \Diamond_1(A_1 \wedge \Diamond_2(A_2 \wedge \dots \wedge \Diamond_n(A_n \wedge \neg q \wedge Hq))) \dots \in \Delta$,
where $\Diamond_i$ is either P or F .

We say that a theory Δ is complete iff for all formulas A , $A \in \Delta$ iff $\neg A \notin \Delta$. The following lemma plays the part of the Lindenbaum lemma in allowing us to work with maximal consistent IRR-theories.

LEMMA 9. *Let A be any Z -consistent formula. Then there exists a complete, Z -consistent IRR-theory, Δ , such that $A \in \Delta$. In fact, if Δ_0 is any Z -consistent theory such that an infinite number of atomic propositions q_i do not appear in Δ_0 , then there exists an IRR Z -consistent and complete theory $\Delta \supseteq \Delta_0$.*

Note that we are only proving a weak completeness result as we need to have a large number of spare atoms.

The main work of the truth lemma of the completeness proof is done by the following lemma in which we say that $\Delta < \Gamma$ iff for all $GA \in \Delta$, $A \in \Gamma$.

LEMMA 10. *Let Δ be a Z -consistent complete IRR-theory. Let $FA \in \Delta$ ($PA \in \Delta$ respectively). Then there exists a Z -consistent complete IRR-theory Γ such that $A \in \Gamma$ and $\Delta < \Gamma$ ($\Delta > \Gamma$ respectively).*

We can finish the completeness proof by then constructing a model from the consistent complete IRR-theories in the usual way: i.e. as worlds in our structure we use all those theories which are connected by some finite amount of $<$ zig-zagging with Δ_0 which contains our formula of interest. The frame of such sets under $<$ is made a structure by making an atom p true at the point Δ iff $p \in \Delta$. So far this is the usual Henkin construction as seen in [Burgess, 2001] for example. The frame will automatically be irreflexive because every set contains an atom which is true there for the first time. So there is no need for bulldozing. Transitivity and linearity follow from the appropriate axioms.

Now the IRR rule, as well as making the completeness proof easier, also arguably makes proving from a set of axioms easier. This is because, being able to consistently introduce names for points into an axiomatic proof

makes the temporal system more like the perhaps more intuitive first-order one. There are none of the problems such as with losing track of “now”.

So given all these recommendations one is brought back to the question of why is it useful to do away with IRR. The growing body of theoretical work (see for example Venema [1991; 1993]) trying to formalize conditions under which the orthodox (to use Venema’s term) system of rules needs to be augmented by something like IRR can be justified as follows:

- adding a new rule of inference to the usual temporal ones is arguably a much more drastic step than adding axioms and it is always important to question whether such additions are necessary;
- in making an unorthodox derivation one may need to go beyond the original language in order to prove a theorem, which makes such axiomatizations less attractive from the point of view of ‘resource awareness’;
- (as argued in [Venema, 1991]), using an atom to perform the naming task of an individual variable in predicate logic is not really in the spirit of temporal/modal logic;

and

- (also as mentioned in [Venema, 1991]), unorthodox axiomatizations do not have some of the nice mathematical properties that orthodox systems have.

2.3 Decidability of US/LT

We have seen that it is often useful to be able to approach the question of consequence in temporal logics in a syntactic way. For many purposes it is enough to be able to determine validity as this is equivalent to determining consequence between finite sets of formulas. A *decision procedure* for a logic is an algorithm for determining whether any given formula is valid or not. The procedure must give correct “yes” or “no” answers for each formula of the language. A logic is said to be *decidable* iff there exists such a decision procedure for its validities.

Notice that a decision procedure is able to tell us more about validities than a complete axiom system. The decision procedure can tell us when a formula is not a validity while an axiom system can only allow us to derive the validities. This is important for many applications.

For many of the most basic temporal logics some general results allow us to show decidability. The logic US/LT is such a logic.

A traditional way of showing decidability for temporal logics is via the so-called *finite model property*. We say that a logic has a finite model property iff any satisfiable formula is satisfiable in a finite model, a model with a

finite number of time points. A systematic search through all finite models coupled with a systematic search through all validities from a complete axiom system gives us a decision procedure. See [Burgess, 2001] for further details of the finite model approach to decidability: this can not be used with US/LT as there are formulas, such as $U(\top, \perp) \wedge GU(\top, \perp)$ which are satisfiable in infinite models only.

The logic US/LT can be shown to be decidable using the translation $*$ of section 2.1 into the first-order monadic logic of order. In [Ehrenfeucht, 1961] it was shown that the first-order logic of linear order is decidable. Other proofs in [Gurevich, 1964] and [Läuchli and Leonard, 1966] show that this also applies to the first-order monadic logic of linear order. The decidability of US/LT follows immediately via lemma 1 and the effectiveness of $*$.

An alternative proof uses the famous result in [Rabin, 1969] showing the decidability of a second-order monadic logic. The logic is $S2S$, the second-order logic of two successors. The language has two unary function symbols l and r as well as a countably infinite number of monadic predicate symbols $P_1, P_2, \dots$. The formulas are interpreted in the binary tree structure (T, l, r) of all finite sequences of zeros and ones with:

$$l(a) = a^\wedge 0, r(a) = a^\wedge 1 \text{ for all } a \in T.$$

As usual a sentence of the language is a formula with no free variables. Rabin shows that $S2S$ is decidable: i.e. there is an algorithm which given a sentence of $S2S$, correctly decides whether or not the sentence is true of the binary tree structure. Proofs of Rabin's difficult result use tree automata.

Rabin's is a very powerful decidability result and much used in establishing the decidability of other logics. For example, Rabin uses it to show that the full monadic second-order theory of the rational order is decidable. That is, there is an algorithm to determine whether a formula in the full monadic second-order language of order (as defined in section 2.1) is true of the order $(\mathbb{Q}, <)$. A short argument via the downward Löwenheim-Skolem theorem (see [Gurevich, 1964] or [Gabbay *et al.*, 1994]) then establishes that the universal monadic second-order theory of the class of all linear flows of time is decidable. Thus US/LT is too.

Once a decision procedure is known to exist for a useful logic it becomes an interesting problem to develop an efficient decision procedure for it. That is an algorithm which gives the “yes” or “no” answers to formulas. We might want to know about the fastest possible such algorithms, i.e. the complexity of the decision problem. To be more precise we need to turn the decision problem for a logic into a question for a Turing machine. There is a particular question about the symbolic representation of atomic propositions since we allow them to be chosen from an infinite set of atoms. A careful approach (seen in a similar example in [Hopcroft and Ullman, 1979]) is to suppose (by renaming) that the propositions actually used in a particular formula are $p_1, \dots, p_n$ and to code p_i as the symbol p followed by i written

in binary. Of course this means that the input to the machine might be a little longer than the length of the formula. In fact a formula of length n may correspond to an input of length about $n \log_2 n$. There is no problem with output: we either want a “1” for, “yes, the formula is a validity” or a “0” for, “no, the formula is not a validity”.

Once we have a rigorously defined task for a Turing machine then we can ask all the usual questions about which complexity classes e.g., P, NP, PSPACE, EXPTIME, etc the problem belongs to. A further, very practical question then arises in the matter of actually describing and implementing efficient decision procedures for the logic. We return briefly to such questions later in the chapter.

For US/LT the complexity is an open problem. A result in [Reynolds, 1999] shows that it is PSPACE-hard. Essentially we can encode the running of a polynomially space-bounded Turing Machine in the logic. It is also believed that the decision problem for the logic is in PSPACE but, so far, this is just conjecture. The procedures which are contained within the decidability proofs above are little help as the method in [Läuchli and Leonard, 1966] relies on an enumeration of the validities in the first-order logic (with no clear guide to its complexity) and the complexity of Rabin’s original procedure is non-elementary.

2.4 Other flows of time

We have had a close look at the logic US/LT . There are many other temporal logics. We can produce other logics by varying our language and its semantics (as we will see in subsection 2.5 below) and we can produce other logics by varying the class of structures which we use to define validity. Let $\mathcal{K}$ be some class of linear orders. We define the $L(U, S)$ (or Kamp) logic over $\mathcal{K}$ to have validities exactly those formulas A of $L(U, S)$ which are true at all points t from any structure $(T, <, h)$ where $(T, <) \in \mathcal{K}$.

For example, the formula $\neg U(\top, \perp)$ is a validity of the $L(U, S)$ logic over the class of all dense linear orders. Let us have a closer look at this logic. In fact we can completely axiomatize this logic by adding the following two axioms to our complete axiomatization of US/LT :

$$\begin{aligned} &\neg U(\top, \perp), \\ &\neg S(\top, \perp), \end{aligned}$$

Soundness is clear. To show completeness, strong completeness, assume that Γ is a maximally consistent set of formulas, and here, this means consistent with the new axioms system. However, Γ will also be consistent with $Ax(U, S)$ and so will have a linear model by theorem 1. Say that $(T, <)$ is linear, $t_0 \in T$, and for all $C \in \Gamma$, $(T, <, h), t_0 \models C$. Now $G\neg U(\top, \perp) \in \Gamma$ because it this formula is a theorem derivable by generalization from one

of the new axioms. Thus, it is clear that no point in the future of t_0 has a discrete successor. Similarly there are no such discrete jumps in the past or immediately on either side of t_0 . Thus $(T, <, h)$ was a dense model all along.

Decidability of the $L(U, S)$ logic over dense time follows almost directly from the decidability of $L(U, S)$ over linear time: to decide A over dense time just decide

$$A \wedge \neg U(\top, \perp) \wedge G\neg U(\top, \perp) \wedge \neg S(\top, \perp) \wedge H\neg S(\top, \perp).$$

A more specific logic still is the $L(U, S)$ logic of rational (numbers) time. Here we define validity via truth at all points in any structure $(\mathbb{Q}, <, g)$ (where $<$ is the usual irreflexive ordering on the rationals). Such a logic has uses in reasoning about events and states when it might be inconvenient to assume that time is Dedekind complete. For example, a well situated gap in time could save arguments about whether there is a last moment when a light is on or a first moment when it is off. To axiomatize this logic it is enough to add to the system for dense time axioms asserting that there is neither an end of time nor a beginning:

$$GF\top, HP\top.$$

The system is clearly sound. There are two ways to see that it is complete, both using the fact that any countable dense ordering without end points is (isomorphic to) the rationals. One way is to notice that Burgess' construction for a model of a set of formulas consistent with $Ax(U, S)$ does construct a countable one. The other way is to use the downward Löwenheim-Skolem theorem on the monadic translations of the temporal formulas.

The same sorts of moves give us decidability of the $L(U, S)$ logic of the rationals via the decidability over general linear time.

Another useful specific dense logic is the $L(U, S)$ logic over real (numbers) time, sometimes loosely called continuous time temporal logic. This is used in many applications as the real numbers seem to be the right model of time for many situations. Unfortunately the real numbers are not straightforward to describe with temporal axioms. The logic was first axiomatized in [Gabbay and Hodkinson, 1990] using a combination of techniques from [Läuchli and Leonard, 1966], [Burgess and Gurevich, 1985] and [Doets, 1989] to do with definable equivalence classes, the IRR approach to axiomatizing the $L(U, S)$ logics over linear time and expressive completeness ideas which we will see in section 3 below.

The axiomatization in [Gabbay and Hodkinson, 1990] consists of the basic axiom system for $L(U, S)$ logic over general linear time using the IRR rule (see above) plus:

$P\top \wedge F\top$	no end points,
$Fp \rightarrow FFp$	density,
$FGq \wedge F\neg q \rightarrow F(Gq \wedge \neg PGq)$	future Dedekind completeness,
$PHq \wedge P\neg q \rightarrow P(Hq \wedge \neg FHq)$	past Dedekind completeness,

and a new axiom,

$$\begin{aligned}
& F(q \wedge F(q \wedge r \wedge H\neg r)) \wedge U(r, q \rightarrow \neg U(q, \neg q)) \\
& \rightarrow F(K^+q \wedge K^-q \wedge F(r \wedge H\neg r))
\end{aligned}$$

called the SEP rule.

The Dedekind completeness axioms are due to Prior and, as we will see, can be used with F and P logics to capture Dedekind completeness, the property of there being no gaps in the flow of time. In fact, these axioms just ensure definable Dedekind completeness, i.e. that there are no gaps in time in a structure which can be noticed by looking at the truth values of formulas.

The axiom SEP is interesting. Nothing like it is needed to axiomatize continuous temporal logic with only Prior's connectives as the property it captures is not expressible without U or S and hence without K^+ or K^- . SEP is associated with the separability of $\mathbb{R}$, i.e. the fact that it has a dense countable suborder (e.g., the rationals). It says roughly that if a formula is densely true in an interval then there is a point at which the formula is true both arbitrarily soon before and afterwards. That SEP is necessary in the axiom system is shown in [Gabbay and Hodkinson, 1990] when a structure is built in which all substitution instances of the other axioms including the Prior ones are valid while SEP is not. Such structures also show that the $L(U, S)$ logic over the reals is distinct from the amp logic over arbitrary continuous flows of time i.e. those that are dense, Dedekind complete and without end points.

The completeness proof only gives a weak completeness result: i.e. the axiom system allows derivation of all validities but it does not give us the general consequence relation between a possibly infinite set of formulas and a formula. In fact it is impossible to give a strongly complete axiom system for this logic because it is not *compact*: there is an infinite set of formulas which is inconsistent but every finite subset of it is consistent. Here is one example:

$$\Gamma = \{FG\neg p, G\neg K^-p, A_0, A_1, \dots\}$$

where $A_0 = Fp$ and for each n , $A_{n+1} = FA_n$.

The proof relies on building a not necessarily real flowed model M of a given satisfiable formula A , say, and then showing that for each n , there is a real flowed structure which satisfies the same monadic sentences to

quantifier depth n as M does. By choosing n to be one more than the depth of quantifiers in $*A$ one can see that we have a model of A by reasoning about the satisfiability of the monadic sentence $\exists x * A(x)$.

The later axiom system in [Reynolds, 1992] is complete for the $L(U, S)$ logic over the reals and does not use the IRR rule. Instead it adds the following axioms to $Ax(U, S)$:

$$\begin{array}{ll}
 K^+ \top, K^- \top, F \top, P \top & \text{as before,} \\
 U(\top, p) \wedge F \neg p \rightarrow U(\neg p \vee K^+ \neg p, p) & \text{Prior-U,} \\
 S(\top, p) \wedge P \neg p \rightarrow S(\neg p \vee K^- \neg p, p) & \text{Prior-S, and} \\
 K^+ p \wedge \neg K^+(p \wedge U(p, \neg p)) \rightarrow K^+(K^+ p \wedge K^- p) & \text{SEP2}
 \end{array}$$

Prior-U and its mirror image are just versions of the Dedekind completeness axioms and SEP2 is a neater version of SEP also developed by Ian Hodkinson. The proof of completeness is similar to that in [Gabbay and Hodkinson, 1990] but requires quite a bit more work as the “names” produced by the IRR rule during construction are not available to help reason about definable equivalence classes.

The decidability of the $L(U, S)$ logic over real time is also not straightforward to establish. It was proven by two different methods in [Burgess and Gurevich, 1985]. One method uses a variant of a traditional approach: show that a formula that is satisfiable over the reals is also satisfiable over the rationals under a valuation which conforms to a certain definition of “niceness”, show that a formula satisfiable under a “nice” valuation on the rationals is satisfiable over the reals, and show that deciding satisfiability under nice valuations over the rationals is decidable. The other method uses arguments about definable equivalence relations as in the axiomatization above. Both methods use Rabin’s decidability result for $S2S$ and Kamp’s expressive completeness result which we will see in a later section.

The complexity of the decision problem for the $L(U, S)$ logic over the reals is an open problem.

Now let us consider the $L(U, S)$ logics over discrete time. To axiomatize the $L(U, S)$ logic over the integers it is not enough to add the following discreteness and non-endedness axioms to the Burgess system $Ax(U, S)$:

$$U(\top, \perp) \text{ and } S(\top, \perp).$$

In fact, we must add these and Prior-style Dedekind completeness axioms such as:

$$Fp \rightarrow U(p, \neg p)$$

and its mirror image. To prove weak completeness (–it is clear that this logic is not compact–) requires a watered down version of the mechanisms for real numbers time or other ways of finding an integer-flowed model from a model with a definably Dedekind complete valuation over some other countable,

discrete flow without end points. There is a proof in [Reynolds, 1994]. An alternative axiom system in the usual IRR style is probably straightforward to construct. The decidability of this logic follows from the decidability of the full monadic second-order theory of the integers which was proved in [Büchi, 1962]. Again the complexity of the problem is open.

When we turn to natural numbers time we find the most heavily studied temporal logics. This is because of the wide-ranging computer science applications of such logics. However, it is not the $L(U, S)$ logic which is studied here but rather logics like PTL which concentrate on the future and which we will meet in section 7 below. The S connective can be shown to be unnecessary in expressing properties: to see this is a straightforward use of the separation property of the $L(U, S)$ logic over the natural numbers (see section 3 below). Despite this it has been argued (e.g., in [Lichtenstein *et al.*, 1985]) that S can help in allowing natural expression of certain useful properties: it is not necessarily easy or efficient to re-express the property without using S . Thus, axioms systems for the $L(U, S)$ logic over the natural numbers have been presented. In [Lichtenstein *et al.*, 1985], such a complete system is given which is in the style of the axiom systems for the logic PTL which we will meet in section 7 below (and so we will not describe it here). In [Venema, 1991] a different but still complete axioms system is given along with others for $L(U, S)$ logics over general classes of well-orderings. This system is simply $Ax(U, S)$ with axioms for discreteness, Dedekind completeness beginning and no end. Again the completeness proof is subtle because the logic is not compact and there are many different countable, discrete, Dedekind complete orderings with a beginning and no end. The $L(U, S)$ logic over the natural numbers is known to be decidable via monadic logic arguments (via [Büchi, 1962]) and, in [Lichtenstein *et al.*, 1985], a PSPACE decision procedure is given and the problem is shown to be PSPACE-complete.

2.5 Other linear time logics

We have met a variety of temporal logics based on using Kamp's U and S connectives (on top of propositional logic) over various classes of linear orders. Basing a logic on other classes of not necessarily linear orders can also give us useful or interesting logics as we will see in section 4 below. However, there is another way of constructing other temporal logics. For various reasons it might be interesting to build a language using other temporal connectives. We may want the temporal language to more closely mimic a particular natural language with its own ways of representing tense or aspect. We may think that U and S do not allow us to express some important properties. Or we may think that U and S allow us to express too much and so the $L(U, S)$ language is unnecessarily complex to reason with for our particular application.

In the next few sections we will consider temporal logics for reasoning about certain classes of linear flows of time based on a variety of temporal languages. By a temporal language here we will mean a language built on top of propositional logic via the recursive use of one or more temporal connectives. By a temporal connective we will mean a logical connective symbol with a first order table as defined above.

Some of the common connectives include, as well as U and S ,:

- Fp $\exists s > tP(s)$,
 it will sometime be the case that p ;
- Pp $\exists s < tP(s)$,
 it was sometime the case that p ;
- Xp $\exists s > tP(s) \wedge \neg \exists r(t < r < s)$,
 there is a next instant and p will hold then;
- Yp $\exists s < tP(s) \wedge \neg \exists r(s < r < t)$,
 there was a previous instant and p held then.

Note that some (all) of these connectives can be defined in terms of U and S .

A traditional temporal (or modal) logic is that with just the connective F over the class of all linear flows of time. This logic (often with the symbol $\Diamond$ used for F) is traditionally known as K4.3 because it can be completely axiomatized by axioms from the basic modal system K along with an axiom known as 4 (for transitivity) and an axiom for linearity which is not called 3 but usually L . The system includes modus ponens substitution and (future) temporal generalization and the axioms:

$$\begin{aligned} G(p \rightarrow q) &\rightarrow (Gp \rightarrow Gq) \\ Gp &\rightarrow GGp \\ G(p \wedge Gp \rightarrow q) &\vee G(q \wedge Gq \rightarrow p) \end{aligned}$$

where GA is the abbreviation $\neg F\neg A$ in terms of F in this language. The proof of (strong) completeness involves a little bit of rearranging of maximal consistent sets as can be seen in [Burgess, 2001] or [Bull and Segerberg, in this handbook]. The decidability and NP-completeness of the decision problem can be deduced from the result of [Ono and Nakamura, 1980] mentioned shortly.

Adding Prior's past connective P to the language, but still defining consequence over the class of all linear orders results in the basic linear $L(F, P)$ logic which is well described in [Burgess, 2001]. A strongly complete axiom system can be obtained by adding mirror images of the rules and axioms in K4.3.

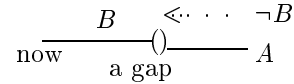
To see that the linear $L(F, P)$ logic is decidable one could simply call on the decidability of the $L(U, S)$ logic over linear time (as seen above). It is a trivial matter to see that a formula in the $L(F, P)$ language can be translated

directly into an equivalent formula in the $L(U, S)$ language. An alternative approach is to show that the $L(F, P)$ logic has a finite model property: if A is satisfiable (in a linear structure) then A is satisfiable in a finite structure (of some type). As described in [Burgess, 2001], in combination with the complete axiom system, this gives an effective procedure for deciding the validity of any formula.

A third alternative is to use the result in [Ono and Nakamura, 1980] that if a $L(F, P)$ formula of length n is satisfiable in a linear model then it is satisfiable in a finite connected, transitive, totally ordered but not necessarily anti-symmetric or irreflexive model containing at most n points. This immediately gives us a non-deterministic polynomial time decision procedure. Since propositional logic is NP-complete we conclude that the linear $L(F, P)$ logic is too.

Another linear time logic has recently been studied in [Reynolds, 1999]. This is the linear time logic with just the connective U . It was studied because, despite the emerging applications of reasoning over general linear time, as we saw above, it is not known how computationally complex it is to decide validity in the linear $L(U, S)$ logic. As a first step to solving this problem the result in this paper shows that the problem of deciding formulas with just U is PSPACE-complete. The proof uses new techniques based on the “mosaics” of [Németi, 1995]. A mosaic-based decision procedure consists in trying to establish satisfiability by guessing and checking a set of model pieces to see if they can be put together to form a model. Mosaics were first used in deciding a temporal logic in [Reynolds, 1998]. It is conjectured that similar methods may be used to show that deciding the $L(U, S)$ logic is also PSPACE-complete.

The logics above have all been obviously not more expressive than the $L(U, S)$ logic of linear time. Are there linear time temporal logics which are more expressive than the $L(U, S)$ logic? We will see later that the answer is yes and that a completely expressive language (in a manner to be defined precisely) contains two more connectives along with Kamp’s. These are the Stavi connectives which were defined in [Gabbay *et al.*, 1980]. $U'(A, B)$ holds if B is true from now until a gap in time after which B is arbitrarily soon false but after which A is true for a while: $U'(A, B)$ is as pictured



S' is defined via the mirror image. Despite involving a gap, U' is in fact a first-order connective. Here is the first-order table for U' :

$$\begin{aligned}
 U'(p, q) \equiv & \\
 \exists s \quad & t < s \\
 \wedge \quad & \forall u \quad (\\
 & \quad ([\quad \exists v(u < v \wedge \forall w(t < w < v \rightarrow q(w)) \quad] \\
 & \quad \vee \quad [\quad \forall v(u < v < s \rightarrow p(v)) \\
 & \quad \quad \wedge \quad \exists v(t < v < u \wedge \neg q(v)) \quad])) \\
 \wedge \quad & \exists u[t < u < s \wedge \neg q(u)] \\
 \wedge \quad & \exists u[t < u < s \wedge \forall v(t < v < u \rightarrow q(v))]
 \end{aligned}$$

Of course, S' has the mirror image table.

We will see in section 3 below that the logic with U , S , U' and S' is expressively complete for the class of structures with linear flow of time. There is no known complete axiom system for the logic with this rather complicated set of connectives. The decidability of the logic follows from the decidability of first-order monadic logic. However, the complexity of the decision problem is also open. If it was shown to be PSPACE-complete for example, then we would have the very interesting result that this temporal logic is far far easier to reason with than the equally expressive monadic logic (with one free variable).

Probably the most useful dense linear time temporal logics are those based on the real-numbers flow of time. Because it is expressively complete (as we will see), the $L(U, S)$ logic over the reals is the most important such logic. The Stavi connectives are useless over the reals. We have seen that this logic can be axiomatized in several ways, and is decidable. However, the complexity of the decision procedure is not known. Other, less expressive, real-flowed temporal logics can be defined. Logics built with any combination of connectives from $\{F, P, U, S\}$ will clearly be decidable. There is an axiomatization of the $L(F, P)$ logic over the reals in [Burgess, 2001].

Various authors have studied a real-flowed temporal logic with a slightly unusual semantics. We say that a structure $(\mathbb{R}, <, h)$ has *finite variability* iff in any bound interval of time, there are only finitely many points of time between which the truth values of all atoms are constant. A logic can be defined by evaluating U and S only on finitely variable structures. This allows the logic to be useful for reasoning about many situations but makes it amenable to the sorts of techniques which are used to reason about sequences of states and natural numbers time temporal logics. See [Kesten *et al.*, 1994] and [Rabinovich, 1998] for more details.

3 THE EXPRESSIVE POWER OF TEMPORAL CONNECTIVES

The expressivity of a language is always measured with respect to some other language. That is, when talking about expressivity, we are always comparing two or more languages. When measuring the expressivity of a large number of languages, it is usually more convenient to have a single

language with respect to which all other languages can be compared, if such a language is known to exist.

In the case of propositional one-dimensional temporal languages defined by the presence of a fixed number of *temporal connectives* (also called *temporal modalities*), the expressivity of those languages can be all measured against a fragment of first-order logic, namely the *monadic first-order language*. This is the fragment that contains a binary $<$ (to represent the underlying temporal order), $=$ (which we assume is always in the language) and a set of unary predicates $Q_1(x), Q_2(x), \dots$ (which account for the interpretation of the propositional letters, that are interpreted as a subset of the temporal domain T). Indeed, any one-dimensional temporal connective can be defined as a well-formed formula in such a fragment, known as the connective's *truth table*; one-dimensionality forces such truth tables to have a single free variable.

In the case of comparing the expressivity of temporal connectives, another parameter must be taken into account, namely the underlying flow of time. Two temporal languages may have the same expressivity over one flow of time (say, the integers) but may differ in expressivity over another (e.g. the rationals); see the discussion on the expressivity of the US connectives below.

Let us exemplify what we mean by those terms. Consider the connectives *since*(S), *until*(U), *future*(F), and *past*(P). Given a flow of time $(T, <, h)$, the truth value of each of the above connectives at a point $t \in T$ is determined as follows:

$$\begin{aligned}
 (T, <, h), t \models Fp & \quad \text{iff} \quad (\exists s > t)(T, <, h), s \models p, \\
 (T, <, h), t \models Pp & \quad \text{iff} \quad (\exists s < t)(T, <, h), s \models p, \\
 (T, <, h), t \models U(p, q) & \quad \text{iff} \quad (\exists s > t)((T, <, h), s \models p \wedge \\
 & \quad \forall y(t < y < s \rightarrow (T, <, h), y \models q)), \\
 (T, <, h), t \models S(p, q) & \quad \text{iff} \quad (\exists s < t)((T, <, h), s \models p \wedge \\
 & \quad \forall y(s < y < t \rightarrow (T, <, h), y \models q))
 \end{aligned}$$

If we assume that $h(p)$ represents a first-order unary predicate that is interpreted as $h(p) \subseteq T$, then these truth values above can be expressed as first-order formulas. Thus:

- (a) $(T, <, h), t \models Fq$ iff $\chi_F(t, h(q))$ holds in $(T, <)$,
- (b) $(T, <, h), t \models Pq$ iff $\chi_P(t, h(q))$ holds in $(T, <)$,
- (c) $(T, <, h), t \models U(q_1, q_2)$ iff $\chi_U(t, h(q_1), h(q_2))$ holds in $(T, <)$, and
- (d) $(T, <, h), t \models S(q_1, q_2)$ iff $\chi_S(t, h(q_1), h(q_2))$ holds in $(T, <)$.

where

- (a) $\chi_F(t, Q) = (\exists s > t)Q(s)$,
- (b) $\chi_P(t, Q) = (\exists s < t)Q(s)$,
- (c) $\chi_U(t, Q_1, Q_2) = (\exists s > t)(Q_1(s) \wedge \forall y(t < y < s \rightarrow Q_2(y)))$,
- (d) $\chi_S(t, Q_1, Q_2) = (\exists s < t)(Q_1(s) \wedge \forall y(s < y < t \rightarrow Q_2(y)))$.

$\chi_\#(t, Q_1, \dots, Q_n)$ is called the *truth table* for the connective $\#$. The number n of parameters in the truth table will be the number of places in the connective, e.g. F and P are one place connective, and their truth tables have a single parameter; S and U are two-place connectives, with truth tables having two parameters.

It is clear that in such a way, we start defining any number of connectives. For example consider $\chi(t, Q) = \exists xy(t < x < y \wedge \forall s(x < s < y \rightarrow Q(s)))$; then $\chi(t, Q)$ means ‘There is an interval in the future of t inside which P is true.’ This is a table for a connective F_{int} : $(T, <, h), t \models F_{\text{int}}(p)$ iff $\chi(t, h(p))$ holds in $(T, <)$.

We are in condition of presenting a general definition of what a temporal connective is:

DEFINITION 11.

1. Any formula $\chi(t, Q_1, \dots, Q_m)$ with one free variable t , in the monadic first-order language with predicate variable symbols Q_i , is called an m -place truth table (in one dimension).
2. Given a syntactic symbol $\#$ for an m -place connective, we say it has a truth table $\chi(t, Q_1, \dots, Q_m)$ iff for any T, h and t , $(*)$ holds:

$$(*) : (T, <, h), t \models \#(q_1, \dots, q_m) \text{ iff } (T, <) \models \chi(t, h(q_1), \dots, h(q_m)).$$

This way we can define as many connectives as we want. Usually, some connectives are definable using other connectives. For example, it is well known that F is definable using U as $Fp \equiv U(p, \top)$. As another example, consider a connective that states the existence of a “next” time point: $\delta \equiv U(\top, \perp)$.

The connective δ is a nice example on how the definability of a connective by others depends on the class of flows of time being considered. For example, in a dense flow of time, δ can be defined in terms of F and P — actually, since there are no “next” time points anywhere, $\delta \equiv \perp$. Similarly, in an integer-like flow of time, δ is equivalent to $\top$.

On the other hand, consider the flow $(T, <)$ of time with a single point without a “next time”: $T = \{\dots - 2, -1, 0, 1, 2, \dots\} \cup \{(1/n) \mid n = 1, 2, 3, \dots\}$, with $<$ being the usual order; then δ is not definable using P and F . To see that, suppose for contradiction that δ is equivalent to A where A is written with P and F and, maybe, atoms. Replace all appearances of atoms by $\perp$

to obtain A' . Since $\delta \leftrightarrow A$ holds in the structure $(T, <, h')$ with all atoms always false, in this structure $\delta \leftrightarrow A'$ holds. As neither δ nor A' contain atoms, $\delta \leftrightarrow A'$ holds in all other $(T, <, h)$ as well. Now A' contains only P and F , $\top$, and $\perp$ and the classical connectives. Since $F\top \equiv P\top \equiv \top$ and $F\perp \equiv P\perp \equiv \perp$, at every point, A' must be equivalent (in $(T, <)$) to either $\top$ or $\perp$ and so cannot equal δ which is true at 1 and false at 0. As a consequence, δ is not definable using P and F over linear time.

In general, given a family of connectives, e.g. $\{F, P\}$ or $\{U, S\}$, we can build new connectives using the given ones. That these new connectives are connectives in the sense of Definition 11 follows from the following.

LEMMA 12. *Let $\#_1(q_1, \dots, q_{m_1}), \dots, \#_n(q_1, \dots, q_{m_n})$ be n temporal connectives with tables $\chi_1, \dots, \chi_n$. Let A be any formula built up from atoms $q_1, \dots, q_m$, the classical connectives, and these connectives. Then there exists a monadic $\psi_A(t, Q_1, \dots, Q_m)$ such that for all T and h ,*

$$(T, <, h), t \models A \text{ iff } (T, <) \models \psi_A(t, h(q_1), \dots, h(q_m)).$$

Proof. We construct ψ_A by induction on A . The simple cases are: $\psi_{q_j} = Q_j(t)$, $\psi_{\neg A} = \neg \psi_A$ and $\psi_{A \wedge B} = \psi_A \wedge \psi_B$.

For the temporal connective case, we construct the formula $\psi_{\#_i(A_1, \dots, A_{m_i})} = \chi_i(t, \psi_{A_1}, \dots, \psi_{A_{m_i}})$; the right-hand side is a notation for the formula obtained by substituting $\psi_{A_j}(x)$ in χ_i wherever $Q_j(x)$ appears, with the appropriate renaming of bound variables to avoid clashes. The induction hypothesis is applied over $\psi_{A_1}, \dots, \psi_{A_{m_i}}$ and the result is simply obtained by truth table of the connective $\#_i$. ■

The formula ψ_A built above is called the *first-order translation* of a temporal formula A . An m -place connective $\#$ with truth table $\chi(t, Q_1, \dots, Q_m)$ is said to be *definable* from connectives $\#_1, \dots, \#_n$ in a flow of time $(T, <)$ if there exists a temporal formula A built from those connectives whose first order translation is ψ_A such that

$$(T, <) \models \psi_A \leftrightarrow \chi.$$

The *expressive power* of a family of connectives over a flow of time is measured by how many connectives it can express over the flow of time. If it can express any conceivable connective (given by a monadic formula), then that family of connectives is expressively complete.

DEFINITION 13. A temporal language with one-dimensional connectives is said to be *expressively complete* or, equivalently, *functionally complete*, in one dimension over a class $\mathcal{T}$ of partial orders iff for any monadic formula $\psi(t, Q_1, \dots, Q_m)$, there exists an A of the language such that for any $(T, <)$ in $\mathcal{T}$, for any interpretation h for $q_1, \dots, q_m$,

$$(T, <) \models \forall t(\psi \leftrightarrow \psi_A)(t, h(q_1), \dots, h(q_m)).$$

In the cases where $\mathcal{T} = \{(T, <)\}$ we talk of expressive completeness over $(T, <)$. For example, the language of Since and Until is expressively complete over integer time and real number flow of time, as we are going to see in Section 3.2; but they are not expressively complete over rational numbers time [Gabbay *et al.*, 1980].

DEFINITION 14. A flow of time $(T, <)$ is said to be expressively complete (or functionally complete) (in one dimension) iff there exists a *finite* set of (one-dimensional) connectives which is expressively complete over $(T, <)$, in one dimension.

The qualification of one-dimensionality in the definitions above will be explained when we introduce the notion of H-dimension below.

These notions parallel the definability and expressive completeness of classical logic. We know that in classical logic $\{\neg, \rightarrow\}$ is sufficient to define all other connectives. Furthermore, for any n -place truth table $\psi : 2^n \rightarrow 2$ there exists an $A(q_1, \dots, q_n)$ of classical logic such that for any h ,

$$h(A) = \psi(h(q_1), \dots, h(q_n)).$$

This is the expressive completeness of $\{\neg, \rightarrow\}$ in classical logic.

The notion of expressive completeness leads us to formulate two questions:

- (a) Given a finite set of connectives and a class of flows of time, are these connectives expressively complete?
- (b) In case the answer to (a) is no, we would like to ask: given a class of flows, does there exist a finite set of one-dimensional connectives that is expressively complete?

These questions occupy us to the rest of this section. We show that the notion of expressive completeness is intimately related to the *separation property*.

The answer to question (b) is related to the notion of H-dimension, discussed in Section 3.3.

3.1 Separation and Expressive Completeness

The notion of separation involves partitioning a flow of time in disjoint parts (typically: present, past and future). A formula is separable if it is equivalent to another formula whose temporal connectives refer only to one of the partitions.

If every formula in a language is separable, that means that we have at least one connective that has enough expressivity over each of the partitions. So we might expect that that set of connectives is expressively complete over

a class of flows that admits such partitioning, provided the partitioning is also expressible by the connectives.

The notion of separation was initially analysed in terms of linear flows, where the notion of present, past and future most naturally applies. So we start our discussion with separation over linear time. We later extend separation to generic flows.

Separation over linear time

Consider a linear flow of time $(T, <)$. Let h, h' be two assignments and $t \in T$. We say that h, h' *agree on the past of t* , $h =_{<t} h'$, iff for any atom q and any $s < t$,

$$s \in h(q) \text{ iff } s \in h'(q).$$

We define $h' =_{=t} h$ for *agreement of the present*, iff for any atom q

$$t \in h(q) \text{ iff } t \in h'(q).$$

and $h' =_{>t} h$, for *agreement on the future*, iff for any atom q and any $s > t$,

$$s \in h(q) \text{ iff } s \in h'(q).$$

Let $\mathcal{T}$ be a class of linear flows of time and A be a formula in a temporal language over $(T, <)$. We say that A is a *pure past formula over $\mathcal{T}$* , iff for all $(T, <)$ in $\mathcal{T}$, for all $t \in T$,

$$\forall h, h', (h =_{<t} h') \text{ implies that } (T, <, h), t \models A \text{ iff } (T, <, h'), t \models A.$$

Similarly, we define *pure future* and *pure present* formulas.

Such a definition of purity is a semantic one. In a temporal language containing only S and U there is also have a notion of syntactic purity as follows. A formula is a *Boolean combination* of $\phi_1, \dots, \phi_n$ if it is built from $\phi_1, \dots, \phi_n$ using only Boolean connectives. A *syntactically pure present* formula is a Boolean combination of atoms only. A *syntactically pure past* formula is a Boolean combination of formulas of the form $S(A, B)$ where A and B are either pure present or pure past. Similarly, a *syntactically pure future* formula is a Boolean combination of formulas of the form $U(A, B)$ where A and B are either pure present or pure future.

It is clear that if A is a syntactically pure past formula, then A is a pure past formula; similarly for pure present and pure future formulas. The converse, however, is not true. For example, from the semantical definition, all temporal temporally valid formulas are pure future (and pure past, and pure present), including those involving S .

We are now in a position to define the separation property.

DEFINITION 15. Let $\mathcal{T}$ be a class of linear flows of time and A be a formula in a temporal language $\mathbf{L}$. We say A is *separable* in $\mathbf{L}$ over $\mathcal{T}$ iff there exists

a formula in $\mathbf{L}$ which is a Boolean combination of pure past, pure future, and atomic formulas and is equivalent to A everywhere in any $(T, <)$ from $\mathcal{T}$.

A set of temporal connectives is said to have the separation property over $\mathcal{T}$ iff every formula in the temporal language of these connectives is separable in the language (over $\mathcal{T}$).

We now show that separation implies expressive completeness.

THEOREM 16. *Let $\mathbf{L}$ be a temporal language built from any number (finite or infinite) of connectives in which P and F are definable over a class $\mathcal{T}$ of linear flows of time. If $\mathbf{L}$ has the separation property over $\mathcal{T}$ then $\mathbf{L}$ is expressively complete over $\mathcal{T}$.*

Proof. If $\mathcal{T}$ is empty, $\mathbf{L}$ is trivially expressively complete, so suppose not. We have to show that for any $\varphi(t, \overline{Q})$ in the monadic theory of linear order with predicate variable symbols $\overline{Q} = (Q_1, \dots, Q_n)$, there exists a formula $A = A(q_1, \dots, q_n)$ in the temporal language such that for all flows of time $(T, <)$ from $\mathcal{T}$, for all h, t , $(T, <, h), t \models A$ iff $(T, <) \models \varphi(t, h(q_1), \dots, h(q_n))$.

We denote this formula by $A[\varphi]$ and proceed by induction on the depth m of nested quantifiers in φ . For $m = 0$, $\varphi(t)$ is quantifier free. Just replace each appearance of $t = t$ by $\top$, $t < t$ by $\perp$, and each $Q_j(t)$ by q_j to obtain $A[\varphi]$.

For $m > 0$, we can assume $\varphi = \exists x \psi(t, x, \overline{Q})$ where ψ has quantifier depth $\leq m$ (the $\forall$ quantifier is treated as derived).

Assuming that we do not use t as a bound variable symbol in ψ and that we have replaced all appearances of $t = t$ by $\top$ and $t < t$ by $\perp$ then the atomic formulas in ψ which involve t have one of the following forms: $Q_i(t)$, $t < y$, $t = y$, or $y < t$, where y could be x or any other variable letter occurring in ψ .

If we regard t as fixed, the relations $t < y$, $t = y$, $t > y$ become unary and can be rewritten, respectively, as $R_{<}(y)$, $R_{=}(y)$ and $R_{>}(y)$, where $R_{<}$, $R_{=}$ and $R_{>}$ are new unary predicate symbols.

Then ψ can be rewritten equivalently as

$$\psi_0^t(x, \overline{Q}, R_{=}, R_{>}, R_{<}),$$

in which t appears only in the form $Q_i(t)$. Since t is free in ψ , we can go further and prove (by induction on the quantifier depth of ψ) that ψ_0^t can be equivalently rewritten as

$$\psi^t = \bigvee_j [\alpha_j(t) \wedge \psi_j^t(x, \overline{Q}, R_{=}, R_{>}, R_{<})],$$

where

- $\alpha_j(t)$ is quantifier free,

- $Q_i(t)$ appear only in $\alpha_j(t)$ and not at all in ψ_j^t ,
- and each ψ_j^t has quantifier depth $\leq m$.

By the induction hypothesis, there is a formula $A_j = A_j(\bar{q}, r_=:, r_>, r_<)$ in the temporal language such that, for any h, x ,

$$(T, <, h), x \models A_j \text{ iff} \\ (T, <) \models \psi_j^t(x, h(q_1), \dots, h(q_n), h(r_=), h(r_>), h(r_<)).$$

Now let $\Diamond q$ be an abbreviation for a temporal formula equivalent (over $\mathcal{T}$) to $Pq \vee q \vee Fq$ whose existence in $\mathbf{L}$ is guaranteed by hypothesis. Then let $B(\bar{q}, r_=:, r_>, r_<) = \bigvee_j (A[\alpha_j] \wedge \Diamond A_j)$. $A[\alpha_j]$ can be obtained from the quantifier free case.

In any structure $(T, <)$ from $\mathcal{T}$ for any h interpreting the atoms $\bar{q}, r_=:, r_>$ and $r_<$, the following are straightforward equivalences

$$\begin{aligned} (T, <, h), t &\models B \\ (T, <, h), t &\models \bigvee_j (A[\alpha_j] \wedge \Diamond A_j) \\ \bigvee_j ((T, <, h), t &\models A[\alpha_j] \wedge (T, <, h), t \models \Diamond A_j) \\ \bigvee_j (\alpha_j(t) \wedge \exists x ((T, <, h), x &\models A_j)) \\ \bigvee_j (\alpha_j(t) \wedge \exists x \psi_j^t(x, h(q_1), \dots, h(q_n), h(r_=), h(r_>), h(r_<))) & \\ \exists x \bigvee_j (\alpha_j(t) \wedge \psi_j^t(x, h(q_1), \dots, h(q_n), h(r_=), h(r_>), h(r_<))) & \\ \exists x \psi_0^t(x, h(q_1), \dots, h(q_n), h(r_=), h(r_>), h(r_<)). & \end{aligned}$$

Now provided we interpret the r atoms as the appropriate R predicates, i.e.:

- $h^*(r_=:) = \{t\}$,
- $h^*(r_<) = \{s \mid t < s\}$, and
- $h^*(r_>) = \{s \mid s < t\}$,

we obtain

$$(T, <, h^*), t \models B \text{ iff } \exists x \psi(t, x, h^*(q_1), \dots, h^*(q_n)) \text{ iff} \\ \varphi(t, h^*(q_1), \dots, h^*(q_n)).$$

B is almost the $A[\varphi]$ we need except for one problem. B contains, besides the q_i , also three other atoms, $r_=:$, $r_>$, and $r_<$, and equation (*) from Definition 9.1.1 above is valid for any h^* which is arbitrary on the q_i but very special on $r_=:, r_>, r_<$. We are now ready to use the separation property (which we haven't used so far in the proof). We use separation to eliminate $r_=:, r_>, r_<$ from B . Since we have separation B is equivalent to a Boolean combination of atoms, pure past formulas, and pure future formulas.

So there is a Boolean combination $\beta = \beta(\overline{p}_+, \overline{p}_-, p_0)$ such that

$$B \leftrightarrow \beta(\overline{B}_+, \overline{B}_-, B_0),$$

where $B_0(\overline{q}, r_>, r_=: r_<)$ is a combination of atoms, $B_+(\overline{q}, r_>, r_=: r_<)$ are pure future, and $B_-(\overline{q}, r_>, r_=: r_<)$ are pure past formulas.

Finally, $B^* = \beta(B_+^*, B_-^*, B_0^*)$ where

- $B_0^* = B_0(\overline{q}, \perp, \top, \perp)$;
- $B_+^* = B_+(\overline{q}, \top, \perp, \perp)$;
- $B_-^* = B_-(\overline{q}, \perp, \perp, \top)$.

Then we obtain for any h^* ,

$$\begin{aligned} (T, <, h^*), t \models B & \text{ iff } (T, <, h^*), t \models \beta(B_+, B_-, B_0) \\ & \text{ iff } (T, <, h^*), t \models \beta(B_+^*, B_-^*, B_0^*) \\ & \text{ iff } (T, <, h^*), t \models B^*. \end{aligned}$$

Hence

$$(T, <, h^*), t \models B^* \text{ iff } (T, <) \models \varphi(t, h^*(\overline{q})).$$

This equation holds for any h^* arbitrary on $\overline{q}$, but restricted on $r_<, r_>, r_=:$. But $r_<, r_>, r_=:$ do not appear in it at all and hence we obtain that for any h , $(T, <, h), t \models B^*$ iff $(T, <) \models \varphi(t, h^*(\overline{q}))$. So make $A[\varphi] = B^*$ and we are done. $\blacksquare$

The converse is also true: expressive completeness implies separation over linear time. The proof involves using the first-order theory of linear time to first separate a first-order formula over linear time; a temporal formula is translated into the first-order language, where it is separated; expressive completeness is needed then to translate each separated first-order subformula into a temporal formula. Details are omitted, but can be found in [Gabbay *et al.*, 1994].

Generalized Separation

The separation property is not restricted to linear flows of time. In this section we generalize the separation property over any class of flows of time and see that Theorem 16 has a generalised version.

The basic idea is to have some relations that will partition every flow of time in $\mathcal{T}$, playing the role of $<$, $>$ and $=$ in the linear case.

DEFINITION 17. Let $\varphi_i(x, y)$, $i = 1, \dots, n$ be n given formulas in the monadic language with $<$ and let $\mathcal{T}$ be a class of flows of time. Suppose $\varphi_i(x, y)$ partition $\mathcal{T}$, that is, for every t in each $(T, <)$ in $\mathcal{T}$ the sets $T(i, t) = \{s \in T \mid \varphi_i(s, t)\}$ for $i = 1, \dots, n$ are mutually exclusive and $\bigcup_i T(i, t) = T$.

In analogy to the way that F and P represented $<$ and $>$, we assume that for each i there is a formula $\beta_i(t, x)$ such that $\varphi_i(t, x)$ and $\beta_i(t, x)$ are equivalent over $\mathcal{T}$ and β_i is a Boolean combination of some $\varphi_j(x, t)$. Also assume that $<$ and $=$ can be expressed (over $\mathcal{T}$) as Boolean combinations of the φ_i .

Then we have the following series of definitions:

- For any t from any $(T, <)$ in $\mathcal{T}$, for any $i = 1, \dots, n$, we say that truth functions h and h' agree on $T(i, t)$ if and only if $h(q)(s) = h'(q)(s)$ for all s in $T(i, t)$ and all atoms q .
- We say that a formula A is *pure* φ_i over $\mathcal{T}$ if for any $(T, <)$ in $\mathcal{T}$, any $t \in T$ and any two truth functions h and h' which agree on $T(i, t)$, we have

$$(T, <, h), t \models A \text{ iff } (T, <, h'), t \models A.$$

- The logic $\mathbf{L}$ has the *generalized separation property* over $\mathcal{T}$ iff every formula A of $\mathbf{L}$ is equivalent over $\mathcal{T}$ to a Boolean combination of pure formula.

THEOREM 18 (generalized separation theorem). *Suppose the language $\mathbf{L}$ can express over $\mathcal{T}$ the 1-place connectives $\#_i$, $i = 1, \dots, n$, defined by:*

$$(T, <, h), t \models \#_i(p) \quad \text{iff} \quad \exists s \quad \varphi_i(s, t) \text{ holds in } (T, <) \\ \text{and } (T, <, h), s \models p.$$

If has the generalized separation property over a class $\mathcal{T}$ of flows of time then $\mathbf{L}$ is expressively complete over $\mathcal{T}$.

A proof of this result appears in [Amir, 1985]. See also [Gabbay *et al.*, 1994].

The converse does not always hold in the general case, for it depends on the theory of the underlying class $\mathcal{T}$.

A simple application of the generalised separation theorem is the following. Suppose we have a first order language with the binary order predicates $<$, $>$, $=$ with their usual interpretation, and suppose it also contains a *parallel* operator $|$ defined by:

$$x|y =_{def} \neg[(x = y) \vee (x < y) \vee (y < x)].$$

Suppose we have a new temporal connective D , defined by

$$(T, <, h), t \models Dq \text{ iff } \exists x|t \text{ such that } (T, <, h), x \models q.$$

Finally, A is said to be *pure parallel* over a class $\mathcal{T}$ of flows of time iff for all t from any $(T, <)$ from $\mathcal{T}$, for all $h =_t h'$,

$$(T, <, h), t \models A \text{ iff } (T, <, h'), t \models A,$$

where $h =_{|t} h'$ iff $\forall x |t \forall q (x \in h(q) \leftrightarrow x \in h'(q))$.

It is clear what separation means in the context of pure present, past, future, and parallel. It is simple to check that the $<, >, =, |$ satisfy the general separation property and other preconditions for using the generalized separation theorem. Thus that theorem gives immediately the following.

COROLLARY 19. *Let $\mathbf{L}$ be a language with F, P, D over any class of flows of time. If $\mathbf{L}$ has a separation then $\mathbf{L}$ is expressively complete.*

3.2 Expressive Completeness of Since and Until over Integer Time

As an example of the applications of separation to the expressive completeness of temporal language, we are going to sketch the proof of separation of the Since and Until-temporal logic containing over linear time. The full proof can be found in [Gabbay, 1989; Gabbay *et al.*, 1994]. With separation and Theorem 16 we immediately obtain that the connectives S and U are expressively complete over the integers; the original proof of the expressive completeness of S and U over the integers is due to Kamp [Kamp, 1968b].

The basic idea of the separation process is to start with a formula in which S and U may be nested inside each other and through several transformation steps we are going to systematically remove U from inside S and vice-versa. This gives us a syntactical separation which, obviously, implies separation.

As we shall see there are eight cases of nested occurrences of U within an S to worry about. It should be noted that all the results in the rest of this section have dual results for the *mirror images* of the formulas. The *mirror image* of a formula is the formula obtained by interchanging U and S ; for example, the mirror image of $U(p \wedge S(q, r), u)$ is $S(p \wedge U(q, r), u)$.

We start dealing with Boolean connectives inside the scope of temporal operators, with some equivalences over integer flows of time. We say that a formula A is *valid* over a flow of time $(T, <)$ if it is true at all $t \in T$; notation: $(T, <) \models A$

LEMMA 20. *The following formulas (and their mirror images) are valid over integer time:*

- $U(A \vee B, C) \leftrightarrow U(A, C) \vee U(B, C);$
- $U(A, B \wedge C) \leftrightarrow U(A, B) \wedge U(A, C);$
- $\neg U(A, B) \leftrightarrow G(\neg A) \vee U(\neg A \wedge \neg B, \neg A);$
- $\neg U(A, B) \leftrightarrow G(\neg A) \vee U(\neg A \wedge \neg B, B \wedge \neg A).$

Proof. Simply apply the semantical definitions. ■

We now show the eight separation cases involving simple nesting and atomic formulas only.

LEMMA 21. *Let p, q, A , and B be atoms. Then each of the formulas below is equivalent, over integer time, to another formula in which the only appearances of the until connective are as the formula $U(A, B)$ and no appearance of that formula is in the scope of an S :*

1. $S(p \wedge U(A, B), q)$,
2. $S(p \wedge \neg U(A, B), q)$,
3. $S(p, q \vee U(A, B))$,
4. $S(p, q \vee \neg U(A, B))$,
5. $S(p \wedge U(A, B), q \vee U(A, B))$,
6. $S(p \wedge \neg U(A, B), q \vee U(A, B))$,
7. $S(p \wedge U(A, B), q \vee \neg U(A, B))$, and
8. $S(p \wedge \neg U(A, B), q \vee \neg U(A, B))$.

Proof. We prove the first case only; omitting the others. Note that $S(p \wedge U(A, B), q)$ is equivalent to

$$\begin{aligned} & S(p, q) \wedge S(p, B) \wedge B \wedge U(A, B) \\ \vee & [A \wedge S(p, B) \wedge S(p, q)] \\ \vee & S(A \wedge q \wedge S(p, B) \wedge S(p, q), q). \end{aligned}$$

Indeed, the original formula holds at t iff there is $s < t$ and $u > s$ such that p holds at s , A at u , B everywhere between s and u , and q everywhere between s and t . The three disjuncts correspond to the cases $u > t, u = t$, and $u < t$ respectively. Note that we make essential use of the linearity of time. $\blacksquare$

We now know the basic steps in our proof of separation. We simply keep pulling out U s from under the scopes of S s and vice versa until there are no more. Given a formula A , this process will eventually leave us with a *syntactically separated* formula, i.e. a formula B which is a Boolean combination of atoms, formulas $U(E, F)$ with E and F built without using S and formulas $S(E, F)$ with E and F built without using U . Clearly, such a B is separated.

We start dealing with more than one U inside an S . In this context, we call a formula in which U and S do not appear *pure*.

LEMMA 22. *Suppose that A and B are pure formulas and that C and D are such that any appearance of U is as $U(A, B)$ and is not nested under any S s. Then $S(C, D)$ is equivalent to a syntactically separated formula in which U only appears as the formula $U(A, B)$.*

Proof. If $U(A, B)$ does not appear then we are done. Otherwise, by rearrangement of C and D into disjunctive and conjunctive normal form, respectively, and repeated use of Lemma 20 we can rewrite $S(C, D)$ equivalently as a Boolean combination of formulas $S(C_i, D_i)$ with no U appearing. Then the preceding lemma shows that each such Boolean constituent is equivalent to a Boolean combination of separated formulas. Thus we have a separated equivalent. ■

Next let us begin the inductive process of removing U s from more than one S . We present the separation in a crescendo. Each step introduces extra complexity in the formula being separated and uses the previous case as a starting point.

LEMMA 23. *Suppose that A, B , possibly subscripted, are pure formulas. Suppose C, D , possibly subscripted, contain no S . Then E has a syntactically separated equivalent if:*

- *the only appearance of U in E is as $U(A, B)$;*
- *the only appearances of U in E are as $U(A_i, B_i)$;*
- *the only appearances of U in E are as $U(C_i, D_i)$;*
- *E is any U, S formula.*

We omit the proof, referring to [Gabbay *et al.*, 1994, Chapter 10] for a detailed account. But note that since each case above uses the previous one as an induction basis, this process of separation tends to be highly exponential. Indeed, the separated version of a formula can be many times larger than the initial one. We finally have the main results.

THEOREM 24 (separation theorem). *Over the integer flow of time, any formula in the $\{U, S\}$ -language is equivalent to a separated formula.*

Proof. This follows directly from the preceding lemma because, as we have already noted, syntactic separation implies separation. ■

THEOREM 25. *The language $\{U, S\}$ is expressively complete over integer time.*

Proof. This follows from the separation theorem and Theorem 16. ■

Other known separation and expressive completeness results over linear time are [Gabbay *et al.*, 1994]:

- The language $\{U, S\}$ is separable over real time. Indeed, it is separable over any Dedekind complete linear flow of time. As a consequence, it is also expressively complete over such flows.
- The language $\{U, S\}$ is *not* separable over the rationals; as a result, it is not separable over the class of linear flows of time, nor is it expressively complete over such flows.

The problem of $\{U, S\}$ over generic linear flows of time is that they may contain *gaps*. It is possible to define a first order formula that makes a proposition true up until a gap and false afterwards. Such formula, however, cannot be expressed in terms of $\{U, S\}$. So is there an extra set of connectives that is expressively complete over the rationals? The answer in this case is yes, and they are called the Stavi connectives. These are connectives whose truth value depends on the existence of gaps in the flow of time, and therefore are always false over integers or reals. For a detailed discussion on separation in the presence of gaps, please refer to [Gabbay *et al.*, 1994, Chapters 11 and 12].

We remain with the following generic question: given a flow of time, can we find a set of connectives that is expressively complete over it? This is the question that we investigate next.

3.3 *H-dimension*

The notion of *Henkin*- or *H-dimension* involves limiting the number of bound variables employed in first-order formulas. We will see that a *necessary* condition for there to exist a finite set of connectives which is expressively complete over a flow of time is that such flow of time have a finite *H-dimension*.

As for a *sufficient* condition for a finite expressively complete set of connectives, we will see that if *many-dimensional connectives* are allowed, then finite *H-dimension* implies the existence of such finite set of connectives. However, when we consider *one-dimensional connectives* such as *Since* and *Until*, finite *H-dimension* is no longer a sufficient condition.

In fact our approach in this discussion will be based on a *weak many-dimensional logic*. It is *many dimensional* because the truth value of a formula is evaluated at more than one time-point. It is *weak* because atomic formulas are evaluated only at a single time point (called the *evaluation point*), while all the other points are the *reference points*). Such weak many dimensionality allows us to define the truth table of many dimensional systems as formulas in the monadic first-order language, as opposed to a full *m-dimensional* system (in which atoms are evaluated at *m* time points) which would require an *m-adic* language.

An *m-dimensional table for an n-place connective* is a formula of the form $\chi(x_1, \dots, x_m; R_1, \dots, R_n)$, where χ is a formula of the first-order predicate

language, written with symbols from $\{<\} \cup \{R_1, \dots, R_n\}$, where $R_1, \dots, R_n$ are special m -place relation symbols. Without loss of expressivity, we will further assume that each term y_j occurring in $R_i(y_1, \dots, y_m)$ is always a variable.

Fix a temporal system $\mathcal{T}$ whose language contains atoms $q_1, q_2, \dots$, the classical connectives, and the special symbols $\#_1, \dots, \#_j$, standing for $n_1, \dots, n_j$ -place connectives respectively. Let $\chi_1, \dots, \chi_j$ be their m -dimensional $n_1, \dots, n_j$ -place tables respectively.

REMARK 26. Since there are finitely many χ_i to consider, we can further assume that there is $b \geq m$ such that each χ_i is written with variables $x_1, \dots, x_b$ only.

The semantics of m -dimensional formulas is given by:

DEFINITION 27. Let $(T, <)$ be a flow of time. Let h be an assignment into T , i.e. for any atom q , $h(q) \subseteq T$. We define the truth value of each formula A of the language of $\mathcal{T}$ at m indices $a_1, \dots, a_{m-1}, t \in T$ under h , as follows:

1. $(T, <, h), a_1, \dots, a_{m-1}, t \models q$ iff $t \in h(q)$, q atomic.
2. $(T, <, h), a_1, \dots, a_{m-1}, t \models A \wedge B$ iff $(T, <, h), a_1, \dots, a_{m-1}, t \models A$ and $(T, <, h), a_1, \dots, a_{m-1}, t \models B$.
3. $(T, <, h), a_1, \dots, a_{m-1}, t \models \neg A$ iff $(T, <, h), a_1, \dots, a_{m-1}, t \not\models A$.
4. For each i ($1 \leq i \leq j$), $(T, <, h), a_1, \dots, a_{m-1}, t \models \#_i(A_1, \dots, A_{n_i})$ iff $T \models \chi_i(a_1, \dots, a_{m-1}, t, h(A_1), \dots, h(A_{n_i}))$, where

$$h(A_k) =_{\text{def}} \{(t_1, \dots, t_m) \in T^m \mid (T, <, h), t_1, \dots, t_m \models A_k\}.$$

Let L^M denote the monadic language with $<$, first-order quantifiers over elements, and an arbitrary number of monadic predicate symbols Q_i for subsets of T . We will regard the Q_i as predicate (subset) variables, implicitly associated with the atoms q_i . We define the translation of an m -dimensional temporal formula A into a monadic formula δA :

1. If A is an atom q_i , we set $\delta A = (x_1 = x_1) \wedge \dots \wedge (x_{m-1} = x_{m-1}) \wedge Q_i(x_m)$.
2. $\delta(A \wedge B) = \delta A \wedge \delta B$, and $\delta(\neg A) = \neg \delta A$.
3. Let $A = \#_i(A_1, \dots, A_{n_i})$, where $\chi_i(x_1, \dots, x_m; R_1, \dots, R_{n_i})$ is the table of $\#_i$. Since we can always rewrite χ such that all occurrences of $R_k(y_1, \dots, y_m)$ in χ are such that the terms y_i are variables, after a

suitable variable replacement we can write δA using only the variables $x_1, \dots, x_b$ as:

$$\delta A = \chi_i(x_1, \dots, x_m, \delta A_1, \dots, \delta A_{n_i}).$$

Clearly, a simple induction gives us that:

$$(T, <, h), a_1, \dots, a_m \models B \text{ iff } T \models \delta B(a_1, \dots, a_m, h(q_1), \dots, h(q_k)).$$

such that $\delta B(a_1, \dots, a_m, h(q_1), \dots, h(q_k))$ uses only the variables $x_1, \dots, x_b$.

Suppose that $\mathcal{K}$ is a class of flows of time, $\bar{x} = x_1, \dots, x_m$ are variables, and $\bar{Q} = Q_1, \dots, Q_r$ are monadic predicates. If $\alpha(\bar{x}, \bar{Q}), \beta(\bar{x}, \bar{Q})$ are formulas in L^M with free variables $\bar{x}$ and free monadic predicates $\bar{Q}$, we say that α and β are $\mathcal{K}$ -equivalent if for all $T \in \mathcal{K}$ and all subsets $S_1, \dots, S_r \subseteq T$,

$$T \models \forall \bar{x} (\alpha(\bar{x}, S_1, \dots, S_r) \leftrightarrow \beta(\bar{x}, S_1, \dots, S_r)).$$

We say the temporal system $\mathcal{T}$ is *expressively complete over $\mathcal{K}$ in n dimensions* ($1 \leq n \leq m$) if for any $\alpha(x_1, \dots, x_n, \bar{Q})$ of L^M with free variables $x_1, \dots, x_n$, there exists a temporal formula $B(\bar{q})$ of $\mathcal{T}$ built up from the atoms $\bar{q} = q_1, \dots, q_r$, such that $\alpha \wedge \bigwedge_{n < i \leq m} x_i = x_i$ and δB are equivalent in $\mathcal{K}$. In this case, $\mathcal{K}$ is said to be *m-functionally complete* in n dimensions (symbolically, FC_n^m); $\mathcal{K}$ is *functionally complete* if it is FC_1^m for some m .

Finally, we define the *Henkin or H-dimension* d of a class $\mathcal{K}$ of flows as the smallest d such that:

- For any monadic formula $\alpha(x_1, \dots, x_n, Q_1, \dots, Q_r)$ in L^M with free variables among $x_1, \dots, x_n$ and monadic predicates $Q_1, \dots, Q_r$ (with n, r arbitrary), there exists an L^M -formula $\alpha'(x_1, \dots, x_n, Q_1, \dots, Q_r)$ that is $\mathcal{K}$ -equivalent to α and uses no more than d different bound variable letters.

We now show that for any class of flows, finite Henkin dimension is equivalent to functional completeness (FC_1^m for some m).

THEOREM 28. *For any class $\mathcal{K}$ of flows of time, if $\mathcal{K}$ is functionally complete then $\mathcal{K}$ has finite H-dimension.*

Proof. Let $\sigma(\bar{Q})$ be any sentence of L^M . By functional completeness, there exists a $B(\bar{q})$ of $\mathcal{T}$ such that the formulas $x_1 = x_1 \wedge \dots \wedge x_m = x_m \wedge \sigma(\bar{Q})$ and $\delta B(x_1, \dots, x_m, \bar{Q})$ are $\mathcal{K}$ -equivalent. We know that δB is written using variables $x_1, \dots, x_b$ only. Hence the sentence $\sigma^* = \exists x_1 \dots \exists x_m \delta B(x_1, \dots, x_m, \bar{Q})$ has at most b variables, and is clearly $\mathcal{K}$ -equivalent to σ . So every sentence of L^M is $\mathcal{K}$ -equivalent to one with at most b variables. This means that $\mathcal{K}$ has H-dimension at most b , so it is finite. $\blacksquare$

We now show the converse. That is, we assume that the class $\mathcal{K}$ of flows of time has finite H-dimension m . Then we are going to construct a temporal logic that is expressively complete over $\mathcal{K}$ and that is weakly $m+1$ -dimensional (and that is why such proof does not work for 1-dimensional systems: it always constructs a logic of dimension at least 2).

Let us call this logic system **d**. Besides atomic propositions $q_1, q_2, \dots$ and the usual Boolean operators, this system has a set of constants (0-place operators) $C_{i,j}^<$ and $C_{i,j}^=$ and unary temporal connectives Π_i and $\Box_i$, for $0 \leq i, j \leq m$. If h is an assignment such that $(h(q) \subseteq T$ for atomic q , the semantics of **d**-formulas is given by:

1. $(T, <, h), x_0, \dots, x_m \models q$ iff $x_0 \in h(q)$ for q atomic.
2. The tables for $\neg, \wedge$ are the usual ones.
3. $(T, <, h), x_0, \dots, x_m \models C_{i,j}^<$ iff $x_i < x_j$. Similarly we define the semantics of $C_{i,j}^=$. $C_{i,j}^=$ are thus called *diagonal constants*.
4. $(T, <, h), x_0, \dots, x_m \models \Pi_i A$ iff $(T, <, h), x_i, \dots, x_i \models A$. So Π_i “projects” the truth value on the i -th dimension.
5. $(T, <, h), x_0, \dots, x_m \models \Box_i A$ iff $(T, <, h), x_0, \dots, x_{i-1}, y, x_{i+1}, \dots, x_m \models A$ for all $y \in T$. So $\Box_i$ is an “always” operator for the i -th dimension.

LEMMA 29. *Let β be a formula of L^M written only using the variable letters $u_0, \dots, u_m$, and having $u_{i_1}, \dots, u_{i_k}$ free for arbitrary $k \leq m$. Then there exists a temporal formula A of **d** such that for all $h, t_0, \dots, t_m \in T$,*

$$(T, <, h), t_0, \dots, t_m \models A \text{ iff } \mathcal{K}, h \models \beta(t_{i_1}, \dots, t_{i_k}).$$

Proof. By induction on β . Assume first that β is atomic. If β is $u_i < u_j$ let $A = C_{i,j}^<$ if $i \neq j$, and $\perp$ otherwise. Similarly for $u_i = u_j$. If β is $Q(u_i)$, let A be $\Pi_i(q)$.

The classical connectives present no difficulties. We turn to the case where β is $\forall u_i \alpha(u_{i_1}, \dots, u_{i_k})$. By induction hypothesis, let A be the formula corresponding α ; then $\Box_i A$ is the formula suitable for β . ■

We are now in a position to prove the converse of Theorem 28.

THEOREM 30. *For any class $\mathcal{K}$ of flows of time, if $\mathcal{K}$ has finite H-dimension then $\mathcal{K}$ is functionally complete.*

Proof. Let $\beta(u_0)$ be any formula of L^M with one free variable u_0 . As $\mathcal{K}$ has H-dimension m , we can suppose that β is written with variables $u_0, \dots, u_m$. By Lemma 29 there exists an A of **d** such that for any $T \in \mathcal{K}$, $t \in T$, and assignment h into T , $(T, <, h), t, \dots, t \models A$ iff $\mathcal{K}, h \models \beta(t)$. ■

As an application of the results above, we show that the class of partial orders is not functionally complete. For consider the formula corresponding to the statement *there are at least n elements in the order*:

$$\sigma_n = \exists x_1, \dots, x_n \bigwedge_{i \neq j} [(x_i \neq x_j) \wedge \neg(x_i < x_j)].$$

It can be shown that such formula cannot be written with less than n variables (e.g. [Gabbay *et al.*, 1994]). Since we are able to say that there are at least n elements in the order for any finite n , the class partial orders have infinite H-dimension and by Theorem 28 it is not functionally complete.

On the other hand, the reals and the integers must have finite H-dimension, for the $\{U, S\}$ temporal logic is expressively complete over both. Indeed, [Gabbay *et al.*, 1994] shows that it has H-dimension at most 3, and so does the theory of linear order.

4 COMBINING TEMPORAL LOGICS

There is a profusion of logics proposed in the literature for the modelling of a variety of phenomena, and many more will surely be proposed in the future. A great part of those logics deal only with “static” aspects, and the temporal evolution is left out. But eventually, the need to deal with the temporal evolution of a model appears. What we want to avoid is the so called *reinvention of the wheel*, that is, reworking from scratch the whole logic, its language, inference system and models, and reproving all its basic properties, when the temporal dimension is added.

We therefore show here several methods for combining logic systems and we study if the properties of the component systems are *transferred* to their combination. We understand a logic system $\mathcal{L}_L$ as composed of three elements:

- (a) a language $\mathcal{L}_L$, normally given by a set of formation rules generating well formed formulas over a signature and a set of logical connectives.
- (b) An inference system, i.e. a relation, $\vdash_L$, between sets of formulas, represented by $\Delta \vdash_L A$. As usual, $\vdash_L A$ indicates that $\emptyset \vdash_L A$.
- (c) The semantics of formulas over a class $\mathcal{K}$ of model structures. The fact that a formulas A is true of or holds at a model $\mathcal{M} \in \mathcal{K}$ is indicated by $\mathcal{M} \models A$.

Each method for combining logic systems proposes a way of generating the language, inference system and model structures from those of the component system.

The first method presented here adds a temporal dimension $\mathbf{T}$ to a logic system $\mathbf{L}$, called the *temporalisation* of a logic system $\mathbf{T}(\mathbf{L})$, with an automatic way of constructing:

- the language of $\mathbf{T}(\mathbf{L})$;
- the inference system of $\mathbf{T}(\mathbf{L})$; and
- the class of temporal models of $\mathbf{T}(\mathbf{L})$.

We do that in a way that the basic properties of soundness, completeness and decidability are *transferred* from the component logics $\mathbf{T}$ and $\mathbf{L}$ to the combined system $\mathbf{T}(\mathbf{L})$.

If the temporalised logic is itself a temporal logic, we have a two dimensional temporal logic $\mathbf{T}(\mathbf{T}')$. Such a logic is too weak, however, because, by construction, the temporal logic $\mathbf{T}'$ cannot refer to the logic system $\mathbf{T}$. We therefore present the *independent combination* $\mathbf{T} \oplus \mathbf{T}'$ in which two temporal logics are symmetrically combined. As before, the language, inference systems and models of $\mathbf{T} \oplus \mathbf{T}'$, and show that the properties of soundness, completeness and decidability are transferred from $\mathbf{T}$ and $\mathbf{T}'$ to $\mathbf{T} \oplus \mathbf{T}'$.

The independent combination is not the strongest way to combine logics; in particular, the independent combination of two linear temporal logic does not necessarily produce a two-dimensional grid model. So we show how to produce the *full join* of two linear temporal logics $\mathbf{T} \times \mathbf{T}'$, such that all models will be two-dimensional grids. However, in this case we cannot guarantee that the basic properties of $\mathbf{T}$ and $\mathbf{T}'$ are transferred to $\mathbf{T} \times \mathbf{T}'$. In this sense, the independent combination $\mathbf{T} \oplus \mathbf{T}'$ is a *minimal symmetrical combination* of logics that automatically transfers the basic properties. Any further interaction between the logics has to be separately investigated.

As a final way of combining logics, we present methods of combination that are motivated by the study of Labelled Deductive Systems (LDS) [Gabbay, 1996].

All temporal logics considered for combination here are assumed to be linear.

4.1 Temporalising a Logic

The first of the combination methods, known as “adding a temporal dimension to a logic system” or simply “temporalising a logic system”, has been initially presented in [Finger and Gabbay, 1992].

Temporalisation is a methodology whereby an arbitrary logic system $\mathbf{L}$ can be enriched with temporal features from a linear temporal logic $\mathbf{T}$ to create a new, *temporalised* system $\mathbf{T}(\mathbf{L})$.

We assume that the language of temporal system $\mathbf{T}$ is the *US* language and its inference system is an extensions of that of *US*/ $\mathcal{K}_{lin}$, with its corresponding class of temporal linear models $\mathcal{K} \subseteq \mathcal{K}_{lin}$.

With respect to the logic $\mathbf{L}$ we assume it is an extension of classical logic, that is, all propositional tautologies are valid in it. The set $\mathcal{L}_{\mathbf{L}}$ is partitioned in two sets, $BC_{\mathbf{L}}$ and $ML_{\mathbf{L}}$. A formula $A \in \mathcal{L}_{\mathbf{L}}$ belongs to the set of *Boolean*

combinations, BC_L , iff it is built up from other formulas by the use of one of the Boolean connectives $\neg$ or $\wedge$ or any other connective defined only in terms of those; it belongs to the set of *monolithic formula* ML_L otherwise.

If L is not an extension of classical logic, we can simply “encapsulate” it in L' with a one-place symbol $\#$ not occurring in either L or T , such that for each formula $A \in \mathcal{L}_L$, $\#A \in \mathcal{L}_{L'}$, $\vdash_L A$ iff $\vdash_{L'} \#A$ and the model structures of $\#A$ are those of A . Note that $ML_{L'} = \mathcal{L}_{L'}$, $BC_{L'} = \emptyset$.

The alphabet of the temporalised language uses the alphabet of L plus the two-place operators S and U , if they are not part of the alphabet of L ; otherwise, we use $\bar{S}$ and $\bar{U}$ or any other proper renaming.

DEFINITION 31. Temporalised formulas The set $\mathcal{L}_{T(L)}$ of formulas of the logic system L is the smallest set such that:

1. If $A \in ML_L$, then $A \in \mathcal{L}_{T(L)}$;
2. If $A, B \in \mathcal{L}_{T(L)}$ then $\neg A \in \mathcal{L}_{T(L)}$ and $(A \wedge B) \in \mathcal{L}_{T(L)}$;
3. If $A, B \in \mathcal{L}_{T(L)}$ then $S(A, B) \in \mathcal{L}_{T(L)}$ and $U(A, B) \in \mathcal{L}_{T(L)}$.

Note that, for instance, if $\Box$ is an operator of the alphabet of L and A and B are two formulas in $\mathcal{L}_L$, the formula $\Box U(A, B)$ is *not* in $\mathcal{L}_{T(L)}$. The language of $T(L)$ is independent of the underlying flow of time, but not its semantics and inference system, so we must fix a class $\mathcal{K}$ of flows of time over which the temporalisation is defined; if $\mathcal{M}_L$ is a model in the class of models of L , $\mathcal{K}_L$, for every formula $A \in \mathcal{L}_L$ we must have either $\mathcal{M}_L \models A$ or $\mathcal{M}_L \models \neg A$. In the case that L is a temporal logic we must consider a “current time” o as part of its model to achieve that condition.

DEFINITION 32. *Semantics of the temporalised logic.*¹ Let $(T, <) \in \mathcal{K}$ be a flow of time and let $g : T \rightarrow \mathcal{K}_L$ be a function mapping every time point in T to a model in the class of models of L . A model of $T(L)$ is a triple $\mathcal{M}_{T(L)} = (T, <, g)$ and the fact that A is true in $\mathcal{M}_{T(L)}$ at time t is written as $\mathcal{M}_{T(L)}, t \models A$ and defined as:

$$\begin{aligned}
 \mathcal{M}_{T(L)}, t \models A, A \in ML_L & \text{ iff } g(t) = \mathcal{M}_L \text{ and } \mathcal{M}_L \models A. \\
 \mathcal{M}_{T(L)}, t \models \neg A & \text{ iff it is not the case that } \mathcal{M}_{T(L)}, t \models A. \\
 \mathcal{M}_{T(L)}, t \models (A \wedge B) & \text{ iff } \mathcal{M}_{T(L)}, t \models A \text{ and } \mathcal{M}_{T(L)}, t \models B. \\
 \mathcal{M}_{T(L)}, t \models S(A, B) & \text{ iff there exists } s \in T \text{ such that } s < t \text{ and} \\
 & \mathcal{M}_{T(L)}, s \models A \text{ and for every } u \in T, \text{ if} \\
 & s < u < t \text{ then } \mathcal{M}_{T(L)}, u \models B.
 \end{aligned}$$

¹We assume that the a model of T is given by $(T, <, h)$ where h maps time points into sets of propositions (instead of the more common, but equivalent, mapping of propositions into sets of time points); such notation highlights that in the temporalised model each time point is associated to a model of L .

$\mathcal{M}_{T(L)}, t \models U(A, B)$ iff there exists $s \in T$ such that $t < s$ and
 $\mathcal{M}_{T(L)}, s \models A$ and for every $u \in T$, if
 $t < u < s$ then $\mathcal{M}_{T(L)}, u \models B$.

The inference system of $T(L)/K$ is given by the following:

DEFINITION 33. Axiomatisation for $T(L)$ An axiomatisation for the temporalised logic $T(L)$ is composed of:

- The axioms of T/K ;
- The inference rules of T/K ;
- For every formula A in $\mathcal{L}_L$, if $\vdash_L A$ then $\vdash_{T(L)} A$, i.e. all theorems of L are theorems of $T(L)$. This inference rule is called **Persist**.

EXAMPLE 34. Consider classical propositional logic $PL = \langle \mathcal{L}_{PL}, \vdash_{PL}, \models_{PL} \rangle$. Its temporalisation generates the logic system $T(PL) = \langle \mathcal{L}_{T(PL)}, \vdash_{T(PL)}, \models_{T(PL)} \rangle$. It is not difficult to see that the temporalised version of PL over any K is actually the temporal logic $T = US/K$.

If we temporalise over K the one-dimensional logic system US/K we obtain the two-dimensional logic system $T(US) = \langle \mathcal{L}_{T(US)}, \vdash_{T(US)}, \models_{T(US)} \rangle = T^2(PL)/K$. In this case we have to rename the two-place operators S and U of the temporalised alphabet to, say, $\bar{S}$ and $\bar{U}$. Note, however, how weak this logic is, for $\bar{S}$ and $\bar{U}$ cannot occur within the scope of U and S .

In order to obtain a model for $T(US)$, we must fix a “current time”, o_1 , in $\mathcal{M}_{US} = (T_1, <_1, g_1)$, so that we can construct the model $\mathcal{M}_{T(US)} = (T_2, <_2, g_2)$ as previously described. Note that, in this case, the flows of time $(T_1, <_1)$ and $(T_2, <_2)$ need not to be the same. $(T_2, <_2)$ is the flow of time of the upper-level temporal system whereas $(T_1, <_1)$ is the flow of time of the underlying logic which, in this case, happens to be a temporal logic. The satisfiability of a formula in a model of $T(US)$ needs two evaluation points, o_1 and o_2 ; therefore it is a *two-dimensional temporal logic*.

The logic system we obtain by temporalising US -temporal logic is the two-dimensional temporal logic described in [Finger, 1992].

This temporalisation process can be repeated n times, generating an n dimensional temporal logic with connectives U_i, S_i , $1 \leq i \leq n$, such that for $i < j$, U_j, S_j cannot occur within the scope of U_i, S_i .

We analyse now the transfer of soundness, completeness and decidability from T and L to $T(L)$; that is, we are assuming the logics T and L have sound, complete and decidable axiomatisations with respect to their semantics, and we will analyse how such properties transfer to the combined system $T(L)$. It is a routine task to analyse that if the inference systems of T and L are sound, so is $T(L)$. So we concentrate on the proof of transference of completeness.

Completeness

We prove the completeness of $\mathsf{T}(\mathsf{L})/\mathcal{K}$ indirectly by transforming a consistent formula A of $\mathsf{T}(\mathsf{L})$ into $\varepsilon(A)$ and then mapping it into a consistent formula of T . Completeness of $\mathsf{T}/\mathcal{K}$ is used to find a T -model for A^* that is used to construct a model for the original $\mathsf{T}(\mathsf{L})$ formula A .

We first define the transformation and mapping. Given a formula $A \in \mathcal{L}_{\mathsf{T}(\mathsf{L})}$, consider the following sets:

$$\mathit{Lit}(A) = \mathit{Mon}(A) \cup \{\neg B \mid B \in \mathit{Mon}(A)\}$$

$$\mathit{Inc}(A) = \{\bigwedge \Gamma \mid \Gamma \subseteq \mathit{Lit}(A) \text{ and } \Gamma \vdash_{\mathsf{L}} \perp\}$$

where $\mathit{Mon}(A)$ is the set of maximal monolithic subformulae of A . $\mathit{Lit}(A)$ is the set of literals occurring in A and $\mathit{Inc}(A)$ is the set of inconsistent formulas that can be built with those. We transform A into A as: $\varepsilon(A)$:

$$\varepsilon(A) = A \wedge \bigwedge_{B \in \mathit{Inc}(A)} (\neg B \wedge G\neg B \wedge H\neg B)$$

The big conjunction $\mathit{in}\varepsilon(A)$ is a theorem of $\mathsf{T}(\mathsf{L})$, so we have the following lemma.

LEMMA 35. $\vdash_{\mathsf{T}(\mathsf{L})} \varepsilon(A) \leftrightarrow A$

If K is a subclass of linear flows of time, we also have the following property (this is where linearity is used in the proof).

LEMMA 36. *Let $\mathcal{M}_{\mathsf{T}}$ be a temporal model over $\mathcal{K} \subseteq \mathcal{K}_{lin}$ such that for some $o \in T$, $\mathcal{M}_{\mathsf{T}}, o \models \sigma(\Box A)$. Then, for every $t \in T$, $\mathcal{M}_{\mathsf{T}}, t \models \sigma(\Box A)$.*

Therefore, if some subset of $\mathit{Lit}(A)$ is inconsistent, the transformed formula $\varepsilon(A)$ puts that fact in evidence so that, when it is mapped into T , inconsistent subformulae will be mapped into falsity.

Now we want to map a $\mathsf{T}(\mathsf{L})$ -formula into a T -formula. For that, consider an enumeration $p_1, p_2, \dots$, of elements of $\mathcal{P}$ and consider an enumeration $A_1, A_2, \dots$, of formulae in ML_{L} . The *correspondence mapping* $\sigma : \mathcal{L}_{\mathsf{T}(\mathsf{L})} \rightarrow \mathcal{L}_{\mathsf{T}}$ is given by:

$$\begin{aligned} \sigma(A_i) &= p_i \text{ for every } A_i \in ML_{\mathsf{L}}, i = 1, 2, \dots \\ \sigma(\neg A) &= \neg \sigma(A) \\ \sigma(A \wedge B) &= \sigma(A) \wedge \sigma(B) \\ \sigma(S(A, B)) &= S(\sigma(A), \sigma(B)) \\ \sigma(U(A, B)) &= U(\sigma(A), \sigma(B)) \end{aligned}$$

The following is the *correspondence lemma*.

LEMMA 37. *The correspondence mapping is a bijection. Furthermore if A is $\mathsf{T}(\mathsf{L})$ -consistent then $\sigma(A)$ is T -consistent.*

LEMMA 38. *If A is $\mathsf{T}(\mathsf{L})$ -consistent, then for every $t \in T$, $G_A(t) = \{B \in \mathsf{Lit}(A) \mid \mathcal{M}_T, t \models \sigma(B)\}$ is finite and L -consistent.*

Proof. Since $\mathsf{Lit}(A)$ is finite, $G_A(t)$ is finite for every t . Suppose $G_A(t)$ is inconsistent for some t , then there exist $\{B_1, \dots, B_n\} \subseteq G_A(t)$ such that $\vdash_{\mathsf{L}} \bigwedge B_i \rightarrow \perp$. So $\bigwedge B_i \in \mathsf{Inc}(A)$ and $\Box \neg(\bigwedge B_i)$ is one of the conjuncts of $\varepsilon(A)$. Applying Lemma 36 to $\mathcal{M}_T, o \models \sigma(\varepsilon(A))$ we get that for every $t \in T$, $\mathcal{M}_T, t \models \neg(\bigwedge \sigma(B_i))$ but by, the definition of G_A , $\mathcal{M}_T, t \models \bigwedge \sigma(B_i)$, which is a contradiction. ■

We are finally ready to prove the completeness of $\mathsf{T}(\mathsf{L})/\mathcal{K}$.

THEOREM 39 (Completeness transfer for $\mathsf{T}(\mathsf{L})$). *If the logical system L is complete and T is complete over a subclass of linear flows of time $\mathcal{K} \subseteq \mathcal{K}_{lin}$, then the logical system $\mathsf{T}(\mathsf{L})$ is complete over $\mathcal{K}$.*

Proof. Assume that A is $\mathsf{T}(\mathsf{L})$ -consistent. By Lemma 38, we have $(T, <) \in \mathcal{K}$ and associated to every time point in T we have a finite and L -consistent set $G_A(t)$. By (weak) completeness of L , every $G_A(t)$ has a model, so we define the temporalised valuation function g :

$$g(t) = \{\mathcal{M}_{\mathsf{L}}^t \mid \mathcal{M}_{\mathsf{L}}^t \text{ is a model of } G_A(t)\}$$

Consider the model $\mathcal{M}_{\mathsf{T}(\mathsf{L})} = (T, <, g)$ over $\mathcal{K}$. By structural induction over B , we show that for every B that is a subformula of A and for every time point t ,

$$\mathcal{M}_T, t \models \sigma(B) \text{ iff } \mathcal{M}_{\mathsf{T}(\mathsf{L})}, t \models B$$

We show only the basic case, $B \in \mathsf{Mon}(A)$. Suppose $\mathcal{M}_T, t \models \sigma(B)$; then $B \in G_A(t)$ and $\mathcal{M}_{\mathsf{L}}^t \models B$, and hence $\mathcal{M}_{\mathsf{T}(\mathsf{L})}, t \models B$. Suppose $\mathcal{M}_{\mathsf{T}(\mathsf{L})}, t \models B$ and assume $\mathcal{M}_T, t \models \neg\sigma(B)$; then $\neg B \in G_A(t)$ and $\mathcal{M}_{\mathsf{L}}^t \models \neg B$, which contradicts $\mathcal{M}_{\mathsf{T}(\mathsf{L})}, t \models B$; hence $\mathcal{M}_T, t \models \sigma(B)$. The inductive cases are straightforward and omitted.

So, $\mathcal{M}_{\mathsf{T}(\mathsf{L})}$ is a model for A over $\mathcal{K}$ and the proof is finished. ■

Theorem 39 gives us sound and complete axiomatisations for $\mathsf{T}(\mathsf{L})$ over many interesting classes of flows of time, such as the class of all linear flows of time, $\mathcal{K}_{lin}$, the integers, $\mathbb{Z}$, and the reals, $\mathbb{R}$. These classes are, in their T versions, decidable and the corresponding decidability of $\mathsf{T}(\mathsf{L})$ is dealt next.

Note that the construction above is finitistic, and therefore does not itself guarantee that compactness is transferred. However, an important corollary of the construction above is that the temporalised system is a *conservative extension* of both original systems, that is, no new theorem in the language of an original system is provable in the combined system. Formally, L_1 is a *conservative extension* of L_2 if it is an extension of L_2 such that if $A \in \mathcal{L}_{\mathsf{L}_2}$, then $\vdash_{\mathsf{L}_1} A$ only if $\vdash_{\mathsf{L}_2} A$.

COROLLARY 40. *Let L be a sound and complete logic system and T be sound and complete over $\mathcal{K} \subseteq \mathcal{K}_{lin}$. The logic system $\mathsf{T}(\mathsf{L})$ is a conservative extension of both L and T .*

Proof. Let $A \in \mathcal{L}_{\mathsf{L}}$ such that $\vdash_{\mathsf{T}(\mathsf{L})} A$. Suppose by contradiction that $\not\models_{\mathsf{L}} A$, so by completeness of L , there exists a model $\mathcal{M}_{\mathsf{L}}$ such that $\mathcal{M}_{\mathsf{L}} \models \neg A$. We construct a temporalised model $\mathcal{M}_{\mathsf{T}(\mathsf{L})} = (T, <, g)$ by making $g(t) = \mathcal{M}_{\mathsf{L}}$ for all $t \in T$. $\mathcal{M}_{\mathsf{T}(\mathsf{L})}$ clearly contradicts the soundness of $\mathsf{T}(\mathsf{L})$ and therefore that of T , so $\vdash_{\mathsf{L}} A$. This shows that $\mathsf{T}(\mathsf{L})$ is a conservative extension of L ; the proof of extension of T is similar. ■

Decidability

The transfer of decidability is also done using the correspondence mapping σ and the transformation η . Such a transformation is actually computable, as the following two lemmas state.

LEMMA 41. *For any $A \in \mathcal{L}_{\mathsf{T}(\mathsf{L})}$, if the logic system L is decidable then there exists an algorithm for constructing $\varepsilon(A)$.*

LEMMA 42. *Over a linear flow of time, for every $A \in \mathcal{L}_{\mathsf{T}(\mathsf{L})}$,*

$$\vdash_{\mathsf{T}(\mathsf{L})} A \text{ iff } \vdash_{\mathsf{T}} \sigma(\varepsilon(A)).$$

Decidability is a direct consequence of these two lemmas.

THEOREM 43. *If L is a decidable logic system, and T is decidable over $\mathcal{K} \subseteq \mathcal{K}_{lin}$, then the logic system $\mathsf{T}(\mathsf{L})$ is also decidable over $\mathcal{K}$.*

Proof. Consider $A \in \mathcal{L}_{\mathsf{T}(\mathsf{L})}$. Since L is decidable, by Lemma 41 there is an algorithmic procedure to build $\varepsilon(A)$. Since σ is a recursive function, we have an algorithm to construct $\sigma(\varepsilon(A))$, and due to the decidability of T over $\mathcal{K}$, we have an effective procedure to decide if it is a theorem or not. Since $\mathcal{K}$ is linear, by Lemma 42 this is also a procedure for deciding whether A is a theorem or not. ■

4.2 Independent Combination

We now deal with the combination of two temporal logic systems. One of them will be called the *horizontal* temporal logic US , while the other will be the *vertical* temporal logic $\bar{\mathsf{U}}\bar{\mathsf{S}}$. If we temporalise the horizontal logic with the vertical logic, we obtain a very weakly expressive system; if US is the internal (horizontal) temporal logic in the temporalisation process (F is derived in US), and $\bar{\mathsf{U}}\bar{\mathsf{S}}$ is the external (vertical) one ($\bar{F}$ is defined in $\bar{\mathsf{U}}\bar{\mathsf{S}}$), we cannot express that vertical and horizontal future operators commute,

$$F\bar{F}A \leftrightarrow \bar{F}FA.$$

In fact, the subformula $F\bar{F}A$ is not even in the temporalised language of $\bar{U}\bar{S}(US)$, nor is the whole formula. In other words, the interplay between the two-dimensions is not expressible in the language of the temporalised $\bar{U}\bar{S}(US)$.

The idea is then to define a method for combining temporal logics that is symmetrical. As usual, we combine the languages, inference systems and classes of models.

DEFINITION 44. Let $Op(L)$ be the set of non-Boolean operators of a generic logic L . Let $\bar{T}$ and T be logic systems such that $Op(T) \cap Op(\bar{T}) = \emptyset$. The *fully combined language* of logic systems $\bar{T}$ and T over the set of atomic propositions $\mathcal{P}$, is obtained by the union of the respective set of connectives and the union of the formation rules of the languages of both logic systems.

Let the operators U and S be in the language of US and $\bar{U}$ and $\bar{S}$ be in that of $\bar{U}\bar{S}$. Their fully combined language over a set of atomic propositions $\mathcal{P}$ is given by

- every atomic proposition is in it;
- if A, B are in it, so are $\neg A$ and $A \wedge B$;
- if A, B are in it, so are $U(A, B)$ and $S(A, B)$.
- if A, B are in it, so are $\bar{U}(A, B)$ and $\bar{S}(A, B)$.

Not only are the two languages taken to be independent of each other, but the set of axioms of the two systems are supposed to be disjoint; so we call the following combination method the *independent combination* of two temporal logics.

DEFINITION 45. Let US and $\bar{U}\bar{S}$ be two US -temporal logic systems defined over the same set $\mathcal{P}$ of propositional atoms such that their languages are independent. The *independent combination* $US \oplus \bar{U}\bar{S}$ is given by the following:

- The fully combined language of US and $\bar{U}\bar{S}$.
- If $(\Sigma, \mathcal{I})$ is an axiomatisation for US and $(\bar{\Sigma}, \bar{\mathcal{I}})$ is an axiomatisation for $\bar{U}\bar{S}$, then $(\Sigma \cup \bar{\Sigma}, \mathcal{I} \cup \bar{\mathcal{I}})$ is an axiomatisation for $US \oplus \bar{U}\bar{S}$. Note that, apart from the classical tautologies, the set of axioms Σ and $\bar{\Sigma}$ are supposed to be disjoint, but not the inference rules.
- The class of independently combined flows of time is $\mathcal{K} \oplus \bar{\mathcal{K}}$ composed of biordered flows of the form $(\tilde{T}, <, \bar{>})$ where the connected components of $(\tilde{T}, <)$ are in $\mathcal{K}$ and the connected components of $(\tilde{T}, \bar{>})$ are in $\bar{\mathcal{K}}$, and $\tilde{T}$ is the (not necessarily disjoint) union of the sets of time points T and $\bar{T}$ that constitute each connected component.

A model structure for $US \oplus \bar{U}\bar{S}$ over $\mathcal{K} \oplus \bar{\mathcal{K}}$ is a 4-tuple $(\tilde{T}, <, \bar{<}, g)$, where $(\tilde{T}, <, \bar{<}) \in \mathcal{K} \oplus \bar{\mathcal{K}}$ and g is an assignment function $g : \tilde{T} \rightarrow 2^{\mathcal{P}}$.

The semantics of a formula A in a model $\mathcal{M} = (\tilde{T}, <, \bar{<}, g)$ is defined as the union of the rules defining the semantics of $US/\mathcal{K}$ and $\bar{U}\bar{S}/\bar{\mathcal{K}}$. The expression $\mathcal{M}, t \models A$ reads that the formula A is true in the (combined) model $\mathcal{M}$ at the point $t \in \tilde{T}$. The semantics of formulas is given by induction in the standard way:

$$\begin{aligned}
\mathcal{M}, t \models p & \quad \text{iff } p \in g(t) \text{ and } p \in \mathcal{P}. \\
\mathcal{M}, t \models \neg A & \quad \text{iff it is not the case that } \mathcal{M}, t \models A. \\
\mathcal{M}, t \models A \wedge B & \quad \text{iff } \mathcal{M}, t \models A \text{ and } \mathcal{M}, t \models B. \\
\mathcal{M}, t \models S(A, B) & \quad \text{iff there exists an } s \in \tilde{T} \text{ with } s < t \text{ and } \mathcal{M}, s \models A \\
& \quad \text{and for every } u \in \tilde{T}, \text{ if } s < u < t \text{ then } \mathcal{M}, u \models B. \\
\mathcal{M}, t \models U(A, B) & \quad \text{iff there exists an } s \in \tilde{T} \text{ with } t < s \text{ and } \mathcal{M}, s \models A \\
& \quad \text{and for every } u \in \tilde{T}, \text{ if } t < u < s \text{ then } \mathcal{M}, u \models B. \\
\mathcal{M}, t \models \bar{S}(A, B) & \quad \text{iff there exists an } s \in \tilde{T} \text{ with } s \bar{<} t \text{ and } \mathcal{M}, s \models A \\
& \quad \text{and for every } u \in \tilde{T}, \text{ if } s \bar{<} u \bar{<} t \text{ then } \mathcal{M}, u \models B. \\
\mathcal{M}, t \models \bar{U}(A, B) & \quad \text{iff there exists an } s \in \tilde{T} \text{ with } t \bar{<} s \text{ and } \mathcal{M}, s \models A \\
& \quad \text{and for every } u \in \tilde{T}, \text{ if } t \bar{<} u \bar{<} s \text{ then } \mathcal{M}, u \models B.
\end{aligned}$$

The also independent combination of two logics appears in the literature under the names of *fusion* or *join*.

As usual, we will assume that $\mathcal{K}, \bar{\mathcal{K}} \subseteq \mathcal{K}_{lin}$, so $<$ and $\bar{<}$ are transitive, irreflexive and total orders; similarly, we assume that the axiomatisations are extensions of $US/\mathcal{K}_{lin}$.

The temporalisation process will be used as an inductive step to prove the transference of soundness, completeness and decidability for $US \oplus \bar{U}\bar{S}$ over $\mathcal{K} \oplus \bar{\mathcal{K}}$. We define the *degree alternation* of a $(US \oplus \bar{U}\bar{S})$ -formula A for US , $dg(A)$:

$$\begin{aligned}
dg(p) &= 0 \\
dg(\neg A) &= dg(A) \\
dg(A \wedge B) &= dg(S(A, B)) = dg(U(A, B)) = \max\{dg(A), dg(B)\} \\
dg(\bar{U}(A, B)) &= dg(\bar{S}(A, B)) = 1 + \max\{\bar{dg}(A), \bar{dg}(B)\}
\end{aligned}$$

and similarly define $\bar{dg}(A)$ for $\bar{U}\bar{S}$.

Any formula of the fully combined language can be seen as a formula of some finite number of alternating temporalisations of the form

$US(\bar{U}\bar{S}(US(\dots)))$; more precisely, A can be seen as a formula of $US(L_n)$, where $dg(A) = n$, $US(L_0) = US$, $\bar{U}\bar{S}(L_0) = \bar{U}\bar{S}$, and $L_{n-2i} = \bar{U}\bar{S}(L_{n-2i-1})$, $L_{n-2i-1} = US(L_{n-2i-2})$, for $i = 0, 1, \dots, \lceil \frac{n}{2} \rceil - 1$.

Indeed, not only the language of $US \oplus \bar{U}\bar{S}$ is decomposable in a finite number of temporalisation, but also its inferences, as the following important lemma indicates.

LEMMA 46. *Let US and $\bar{U}\bar{S}$ be two complete logic systems. Then, A is a theorem of $US \oplus \bar{U}\bar{S}$ iff it is a theorem of $US(L_n)$, where $dg(A) = n$.*

Proof. If A is a theorem of $US(L_n)$, all the inferences in its deduction can be repeated in $US \oplus \bar{U}\bar{S}$, so it is a theorem of $US \oplus \bar{U}\bar{S}$.

Suppose A is a theorem of $US \oplus \bar{U}\bar{S}$; let $B_1, \dots, B_m = A$ be a deduction of A in $US \oplus \bar{U}\bar{S}$ and let $n' = \max\{dg(B_i)\}$, $n' \geq n$. We claim that each B_i is a theorem of $US(L_{n'})$. In fact, by induction on m , if B_i is obtained in the deduction by substituting into an axiom, the same substitution can be done in $US(L_{n'})$; if B_i is obtained by Temporal Generalisation from B_j , $j < i$, then by the induction hypothesis, B_j is a theorem of $US(L_{n'})$ and so is B_i ; if B_i is obtained by Modus Ponens from B_j and B_k , $j, k < i$, then by the induction hypothesis, B_j and B_k are theorems of $US(L_{n'})$ and so is B_i .

So A is a theorem of $US(L_{n'})$ and, since US and $\bar{U}\bar{S}$ are two complete logic systems, by Theorem 39, each of the alternating temporalisations in $US(L_{n'})$ is a conservative extension of the underlying logic; it follows that A is a theorem of $US(L_n)$, as desired. ■

Note that the proof above gives conservativeness as a corollary. The transference of soundness, completeness and decidability also follows directly from this result.

THEOREM 47 (Independent Combination). *Let US and $\bar{U}\bar{S}$ be two sound and complete logic systems over the classes $\mathcal{K}$ and $\bar{\mathcal{K}}$, respectively. Then their independent combination $US \oplus \bar{U}\bar{S}$ is sound and complete over the class $\mathcal{K} \oplus \bar{\mathcal{K}}$. If US and $\bar{U}\bar{S}$ are complete and decidable, so is $US \oplus \bar{U}\bar{S}$.*

Proof. Soundness follows immediately from the validity of axioms and inference rules.

We only sketch the proof of completeness here. Given a $US \oplus \bar{U}\bar{S}$ -consistent formula A , Lemma 46 is used to see that it is also consistent in $US(L_n)$, so a temporalised $US(L_n)$ -model is built for it. Then, by induction on the degree of alternation of A , this $US(L_n)$ is used to construct a $US \oplus \bar{U}\bar{S}$ -model; each step of such construction preserves the satisfiability of formulas of a limited degree of alternation, so in the final model, A , is satisfiable; and completeness is proved. For details, see [Finger and Gabbay, 1996].

For decidability, suppose we want to decide whether a formula $A \in US \oplus \bar{U}\bar{S}$ is a theorem. By Lemma 46, this is equivalent to deciding whether $A \in US(L_n)$ is a theorem, where $n = dg(A)$. Since $US/\mathcal{K}$ and $\bar{U}\bar{S}/\bar{\mathcal{K}}$ are both

complete and decidable, by successive applications of Theorems 39 and 43, it follows that the following logics are decidable: $US(\bar{U}\bar{S})$, $\bar{U}\bar{S}(US(\bar{U}\bar{S})) = \bar{U}\bar{S}(L_2)$, $\dots$, $\bar{U}\bar{S}(L_{n-1}) = L_n$; so a the last application of Theorems 39 and 43 yields that $US(L_n)$ is decidable. $\blacksquare$

The minimality of the independent combination

The logic $US \oplus \bar{U}\bar{S}$ is the minimal logic that conservatively extends both US and $\bar{U}\bar{S}$. This result was first shown for the independent combination of monomodal logics independently by [Kracht and Wolter, 1991] and [Fine and Schurz, 1991].

Indeed, suppose there is another logic T_1 that conservatively extends both US and $\bar{U}\bar{S}$ but some theorem A of $US \oplus \bar{U}\bar{S}$ is not a theorem of T_1 . But A can be obtained by a finite number of inferences $A_1, \dots, A_n = A$ using only the axioms of US and $\bar{U}\bar{S}$. But any conservative extension of US and $\bar{U}\bar{S}$ must be able to derive A_i , $1 \leq i \leq n$, from $A_1, \dots, A_{i-1}$, and therefore it must be able to derive A ; contradiction.

Once we have this minimal combination between two logic systems, any other interaction between the logics must be considered on its own. As an example, consider the following formulas expressing the commutativity of future and past operators between the two dimensions are not generally valid over a model of $US \oplus \bar{U}\bar{S}$:

- $l_1 \quad F\bar{F}A \leftrightarrow \bar{F}FA$
- $l_2 \quad F\bar{P}A \leftrightarrow \bar{P}FA$
- $l_3 \quad P\bar{F}A \leftrightarrow \bar{F}PA$
- $l_4 \quad P\bar{P}A \leftrightarrow \bar{P}PA$

Now consider the *product* of two linear temporal models, given as follows.

DEFINITION 48. Let $(T, <) \in \mathcal{K}$ and $(\bar{T}, \bar{<}) \in \bar{\mathcal{K}}$ be two linear flows of time. The *product* of those flows of time is $(T \times \bar{T}, <, \bar{<})$. A *product model* over $\mathcal{K} \times \bar{\mathcal{K}}$ is a 4-tuple $\mathcal{M} = (T \times \bar{T}, <, \bar{<}, g)$, where $g : T \times \bar{T} \rightarrow 2^{\mathcal{A}}$ is a two-dimensional assignment. The semantics of the horizontal and vertical operators are independent of each other:

$$\begin{aligned} \mathcal{M}, t, x \models S(A, B) \quad &\text{iff} \quad \text{there exists } s < t \text{ such that } \mathcal{M}, s, x \models A \\ &\text{and for all } u, s < u < t, \mathcal{M}, u, x \models B. \\ \mathcal{M}, t, x \models \bar{S}(A, B) \quad &\text{iff} \quad \text{there exists } y \bar{<} x \text{ such that } \mathcal{M}, t, y \models A \\ &\text{and for all } z, y \bar{<} z \bar{<} x, \mathcal{M}, t, z \models B. \end{aligned}$$

Similarly for U and $\bar{U}$, the semantics of atoms and Boolean connectives remaining the standard one. A formula A is valid over $\mathcal{K} \times \bar{\mathcal{K}}$ if for all models $\mathcal{M} = (T, <, \bar{T}, \bar{<}, g)$, for all $t \in T$ and $x \in \bar{T}$ we have $\mathcal{M}, t, x \models A$.

It is easy to verify that the formulas l_1 – l_4 are valid over product models. We wonder if such product of logics transfers the properties we have investigated for the previous logics. The answer is: it depends. We have the following results.

PROPOSITION 49.

- (a) *There is a sound and complete axiomatisation for $US \times \bar{U}\bar{S}$ over the classes of product models $\mathcal{K}_{lin} \times \mathcal{K}_{lin}$, $\mathcal{K}_{dis} \times \mathcal{K}_{dis}$, $\mathbb{Q} \times \mathbb{Q}$, $\mathcal{K}_{lin} \times \mathcal{K}_{dis}$, $\mathcal{K}_{lin} \times \mathbb{Q}$ and $\mathbb{Q} \times \mathcal{K}_{dis}$ [Finger, 1994].*
- (b) *There are no finite axiomatisations for the valid two-dimensional formulas over the classes $\mathbb{Z} \times \mathbb{Z}$, $\mathbb{N} \times \mathbb{N}$ and $\mathbb{R} \times \mathbb{R}$ [Venema, 1990].*

Note that the all the component one-dimensional mentioned above logic systems are complete and decidable, but their product sometimes is complete, sometimes not. Also, the logics in (a) are all decidable and those in (b) are undecidable.

This is to illustrate the following idea: given an independent combination of two temporal logics, the addition of extra axioms, inference rules or an extra condition on its models has to be studied on its own, just as adding a new axiom to a modal logic or imposing a new property on its accessibility relation has to be analysed on its own.

Combinations of logics in the literature

The work on combining temporal logics presented here has first appeared in the literature in [Finger and Gabbay, 1992; Finger and Gabbay, 1996].

General combinations of logics have been addressed in the literature in various forms. Combinations of tense and modality were discussed in the next chapter in this volume, (which reproduces [Thomason, 1984]), without explicitly providing a general methodology for doing so. A methodology for constructing logics of belief based on existing deductive systems is the *deductive model* of Konolige [Konolige, 1986]; in this case, the language of the original system was the base for the construction of a new modal language, and the modal logic system thus generated had its semantics defined in terms of the inferences of the original system. This is a methodology quite different from the one adopted here, in which we separately combine language, inference systems and class of models.

Combination of two monomodal logics and the transference of properties have been studied by Kracht and Wolter [1991] and Fine and Schurz [1991]; the latter even considers the transference of properties through the combination of n -monomodal logics. These works differ from the combination of temporal logics in several ways: their modalities have no interaction whatsoever (unlike S and U , which actually interact with each other); they only

consider one-place modalities ($\Box$); and their constructions are not a recursive application of the temporalisation (or any similar external application of one logic to another).

A stronger combination of logics have been investigated by Gabbay and Shehtman [Gabbay and Shehtman, 1998], where the starting point is the product of two Kripke frames, generating the *product* of the two monomodal logics. It shows that the transference of completeness and decidability can either succeed or fail for the product, depending on the properties of the component logics. The failure of transference of decidability for temporal products in $FP/\mathcal{K}_{lin} \times FP/\mathcal{K}_{lin}$ has been shown in [Marx and Reynolds, 1999], and fresh results on the products of logics can be found in [Reynolds and Zakharyashev, 2001].

The transference of soundness, completeness and decidability are by no means the only properties to study. Kracht and Wolter [Kracht and Wolter, 1991] study the transference of interpolation between two monomodal logics. The complexity of the combination of two monomodal logics is studied in [Spaan, 1993]; the complexity of products are studied in [Marx, 1999]. Gabbay and Shehtman [Gabbay and Shehtman, 1998] report the failure of transference of the finite model property for their product of modal logics. With respect to specific temporal properties, the transference of the separation property is studied in [Finger and Gabbay, 1992].

For a general combining methodology, see [Gabbay, 1998].

5 LABELLED DEDUCTION PRESENTATION OF TEMPORAL LOGICS

5.1 Introducing LDS

This section develops proof theory for temporal logic within the framework of labelled deductive systems [Gabbay, 1996].

To motivate our approach consider a temporal formula $\alpha = FA \wedge PB \wedge C$. This formula says that we want A to hold in the future (of now), B to hold in the past and C to hold now. It represents the following temporal configuration:

- $t < d < s, t \models B, d \models C$ and $s \models A$

where d is now and t, s are temporal points.

Suppose we want to be very explicit about the temporal configuration and say that we want another instance of B to hold in the past of s but not in the past or future of d , i.e. we want an additional point r such that:

- $r < s, \sim (r = d \vee r < d \vee d < r)$ and $r \models B$.

The above cannot be expressed by a formula. The obvious formula $\beta = F(A \wedge PB) \wedge PB \wedge C$ will not do. We need extra expressive power. We can, for example, use an additional atom q and write

$$\gamma = q \wedge Hq \wedge Gq \wedge F(A \wedge P(B \wedge \sim q)) \wedge PB \wedge C.$$

This will do the job.

However, by far the simplest approach is to allow names of points in the language and write $s : A$ to mean that we want $s \models A$ to hold. Then we can write a theory Δ as

$$\Delta = \{t < d < s, t : A, d : C, s : B, r : A, \sim (r < d \vee r = d \vee d < r)\}.$$

Δ is satisfied if we find a model (S, R, a, h) in which d can be identified with $g(d) = a$ and t, s, r can be identified with some points $g(t), g(s), g(r)$ such that the above ordering relations hold and the respective formulae are satisfied in the appropriate points.

The above language has turned temporal logic into a labelled deductive system (LDS). It has brought some of the semantics into the syntax.

But how about proof theory?

Consider $t : FFPB$. This formula does hold at t (because B holds at r). Thus we must have rules that allow us to show that

$$\Delta \vdash FFPB.$$

It is convenient to write Δ as:

Assumptions	Configuration
$t : B$	$t < d < s$
$d : C$	$\sim (r < d \vee d = r \vee d < r)$
$s : A$	$r < s$
$r : B$	

and give rules to manipulate the configuration until we get $t : FFPB$.

Thus formally a temporal database is a set of labelled formulae $\{t_i : A_i\}$ together with a configuration on $\{t_i\}$, given in the form of an earlier-later relation $<$. A query is a labelled formula $t : Q$. The proof rules have the form

$$\frac{t_1 : A_1; \dots; t_n : A_n, \text{ configuration}}{s : B \text{ configuration'}}$$

where the configuration is a set of conditions on the ordering of $\{t_i\}$.

The use of labels is best illustrated via examples:

EXAMPLE 50. This example shows the LDS in the case of modal logic. Modal logic has to do with possible worlds. Thus we think of our basic database (or assumptions) as a finite set of information about possible

worlds. This consists of two parts. The configuration part, the finite configuration of possible worlds for the database, and the assumptions part which tells us what formulae hold in each world. The following is an example of a database:

Assumptions		Configuration
(1)	$t : \Box\Box B$	$t < s$
(2)	$s : \Diamond(B \rightarrow C)$	

The conclusion to show (or query) is

$$t : \Diamond\Diamond C.$$

The derivation is as follows:

- (3) From (2) create a new point r with $s < r$ and get $r : B \rightarrow C$.

We thus have

Assumptions	Configuration
(1), (2), (3)	$t < s < r$

- (4) From (1), since $t < s$ get $s : \Box B$.
 (5) From (4) since $s < r$ get $r : B$.
 (6) From (5) and (3) we get $r : C$.
 (7) From (6) since $s < r$ get $s : \Diamond C$.
 (8) From (7) using $t < s$ we get $t : \Diamond\Diamond C$.

Discussion The object rules involved are:

$\Box E$ rule:

$$\frac{t < s; t : \Box A}{s : A}$$

$\Diamond I$ rule:

$$\frac{t < s, s : B}{t : \Diamond B}$$

$\Diamond E$ rule:

$$\frac{t : \Diamond A}{\text{create a new point } s \text{ with } t < s \text{ and deduce } s : A}$$

Note that the above rules are not complete. We do not have rules for deriving, for example, $\Box A$. Also, the rules are all for intuitionistic modal logic.

The metalevel considerations which determine which logic we are working in, may be properties of $<$, e.g. $t < s \wedge s < r \rightarrow t < r$, or linearity, e.g. $t < s \vee t = s \vee s < t$ etc.

There are two serious problems in modal and temporal theorem proving. One is that Skolem functions for $\exists x \Diamond A(x)$ and $\Diamond \exists x A(x)$ are not logically the same. If we ‘Skolemize’ we get $\Diamond A(c)$. Unfortunately it is not clear where c exists, in the current world ($\exists x = c \Diamond A(x)$) or the possible world ($\Diamond \exists x = c A(x)$).

If we use labelled assumptions then $t : \exists x \Diamond A(x)$ becomes $t : \Diamond A(c)$ and it is clear that c is introduced at t .

On the other hand, the assumption $t : \Diamond \exists x A(x)$ will be used by the $\Diamond E$ rule to introduce a new point $s, t < s$ and conclude $s : \exists x A(x)$. We can further ‘Skolemize’ at s and get $s : A(c)$, with c introduced at s . We thus need the mechanism of remembering or labelling constants as well, to indicate where they were first introduced.

EXAMPLE 51. Another example has to do with the Barcan formula

Assumption		Configuration
(1)	$t : \forall x \Box A(x)$	$t < s$

We show

(2) $s : \forall x A(x)$.

We proceed intuitively

(3) $t : \Box A(x)$ (stripping $\forall x$, remembering x is arbitrary).

(4) Since the configuration contains $s, t < s$ we get

$s : A(x)$.

(5) Since x is arbitrary we get

$s : \forall x A(x)$.

The above intuitive proof can be restricted.

The rule

$$\frac{t : \Box A(x), t < s}{s : A(x)}$$

is allowed only if x is instantiated.

To allow the above rule for arbitrary x is equivalent to adopting the Barcan formula axiom

$$\forall x \Box A(x) \rightarrow \Box \forall x A(x).$$

EXAMPLE 52. To show $\forall x \Box A(x) \rightarrow \Box \forall x A(x)$ in the modal logic where it is indeed true.

- (1) Assume $t : \forall x \Box A(x)$.

We show $\Box \forall x A(x)$ by the use of the metabox:

	create α ,	$t < \alpha$
(2)	$t : \Box A(x)$	from (1)
(3)	$\alpha : A(x)$	from (2) using a rule which allows this with x a variable.
(4)	$\alpha : \forall x A(x)$	universal generalization.

- (5) Exit: $t : \Box \forall x A(x)$.

This rule has the form

Create α ,	$t < \alpha$
Argue to get	$\alpha : B$
Exit with	$t : \Box B$

5.2 LDS semantics

We can now formally define a simplified version of LDS, sufficient for our temporal logics. The reader is referred to [Gabbay, 1996] for full details.

An algebraic LDS is built up from two components: an algebra $\mathcal{A}$ and a logic $\mathbf{L}$. To make things specific, let us assume that we are dealing with a particular algebraic model $\mathcal{A} = (S, <, f_1, \dots, f_k)$, where S is the domain of the algebra, $<$ is a strict order, i.e. irreflexive and transitive, relation on S and $f_1, \dots, f_k$ are function symbols on S of arities $r_1, \dots, r_k$ respectively. The sequence $\Sigma = (<, f_1, \dots, f_k)$ is called the signature of $\mathcal{A}$. It is really the language of $\mathcal{A}$ in logical terms but we use Σ to separate it from $\mathbf{L}$. We assume the functions are *isotonic*, i.e. they are either monotonic up or monotonic down in each variable, namely for each coordinate x in f we have that either

$$\forall x, y (x < y \rightarrow f(\dots, x, \dots) < f(\dots, y, \dots))$$

or

$$\forall x, y (x < y \rightarrow f(\dots, y, \dots) < f(\dots, x, \dots))$$

holds.

A typical algebra is a binary tree algebra where each point x in the tree has two immediate successor points $r_1(x)$ and $r_2(x)$ and one predecessor point $p(x)$. $<$ is the (branching) earlier–later relation and we have $p(x) < x < r_i(x)$, $i = 1, 2$.

The general theory of LDS (see [Gabbay, 1996, Section 3.2]) requires a source of labels and a source of formulae. These together are used to form the declarative units of the form $t : A$, where t is a label and A is a formula.

The labels can be syntactical terms in some algebraic theory. The algebraic theory itself can be characterized either syntactically by giving axioms which the terms must satisfy or semantically by giving a class of models (algebras) for the language.

The formulae part of an LDS declarative unit is defined in the traditional way in some language $\mathbf{L}$.

An LDS database (theory) Δ is a set of terms and their formulae (i.e. a set of declarative units) with some relationships between the terms. In a general LDS, the terms themselves are syntactical and one always has to worry whether the required relations between the terms of Δ are possible (i.e. are consistent).

If, however, we have a semantic theory for the labels characterized by one single model (algebra), then we can take the labels to be elements of this model and consistency and relationships among the labels (elements) of Δ will always be clear—they are as dictated by the model. This represents a temporal logic with a concrete specific flow of time (e.g. integers, rationals, reals, etc.).

We therefore present for the purpose of this chapter, a concrete definition of LDS based on a single model as an algebra of labels.

DEFINITION 53 (Concrete algebraic LDS).

1. A concrete algebraic LDS has the form $(\mathcal{A}, \mathbf{L})$ where:
 - (a) $\mathcal{A} = (S, <, f_1, \dots, f_k)$ is a concrete algebraic model. The elements of S are called labels.
 - (b) $\mathbf{L}$ is a predicate language with connectives $\sharp_1, \dots, \sharp_m$ with arities $r_1, \dots, r_m$, and quantifiers $(Q_1x), \dots, (Q_m'x)$. The connectives can be some well-known modalities, binary conditional, etc., and the quantifiers can be some known generalized or traditional quantifiers. We assume that the traditional syntactical notions for $\mathbf{L}$ are defined. We also assume that $\mathbf{L}$ has only constants, no function symbols and the constants of $\mathbf{L}$ are indexed by elements of the algebra $\mathcal{A}$, i.e. have the form c_i^t , $t \in S$, $i = 1, 2, 3, \dots$
2. A declarative unit has the form $t : A$, where A is a wff and $t \in S$, or it has the form $t : c_i^s$. The unit $t : A$ intuitively means ‘ A holds at label t ’ and the unit $t : c_i^s$ means ‘the element c_i which was created at label s does exist in the domain of label t ’.

3. A database has the form $\Delta = (D, \mathbf{f}, d, U)$ where $D \subseteq S$ is non-empty and $\mathbf{f}$ is a function associating with each $t \in D$ a set of wffs $\mathbf{f}(t) = \Delta_t$. U is a function associating with each $t \in D$ a set of terms U_t . $d \in D$ is a distinguished point in D . The theory Δ can be displayed by writing $\{t : A_1, t : A_2, s : B, r : c_3^s, \dots\}$, where $t : A$ indicates $A \in \mathbf{f}(t)$ and $r : c^s$ indicates $c^s \in U_r$.

DEFINITION 54 (Semantics for LDS). Let $(\mathcal{A}, \mathbf{L})$ be a concrete algebraic LDS, with algebra $\mathcal{A}$ and language $\mathbf{L}$ with connectives $\{\sharp_1, \dots, \sharp_m\}$ and quantifiers $\{Q_1, \dots, Q_{m'}\}$ where $\sharp_i$ is r_i place.

1. A semantical interpretation for the LDS has the form $\mathcal{I} = (\Psi_0(x, X), \Psi_1, \dots, \Psi_i(x, X_1, \dots, X_{r_i}), \dots, \Psi_m, \Psi'_1(x, Z), \dots, \Psi'_{m'}(x, Z))$ where Ψ_i is a formula of the language of $\mathcal{A}$, possibly second order, with the single free element variable x and the free set variables X as indicated, and Ψ'_i have a single free element variable x and free binary relation variable Z . We need to assume that Ψ_i and Ψ'_j have the property that if we substitute in them for the set variables closed under $\leq$ then the element variable coordinate is monotonic up under $\leq$. In symbols:

$$\bullet \bigwedge_j \forall x, y (x \in X_j \wedge x \leq y \rightarrow y \in X_j) \rightarrow [x \leq y \rightarrow \Psi(x, X_j) \rightarrow \Psi(y, X_j)].$$

2. A model for the LDS has the form $\mathbf{m} = (V, h, g, d)$ where $d \in S$ is the distinguished world, V is a function associating a domain V_t with each $t \in S$, h is a function associating with each n -place atomic predicate P a subset $h(t, P) \subseteq V_t^n$.

g is an assignment giving each variable x an element $g(x) \in V_d$ and for each constant c_i^s an element $g(c_i^s) \in V_s$.

3. Satisfaction is defined by structural induction as follows:

- $t \models P(b_1, \dots, b_n)$ iff $(b_1, \dots, b_n) \in h(t, P)$;
- $t \models \exists x A(x)$ iff for some $b \in V_t, t \models A(b)$;
- $t \models A \wedge B$ iff $t \models A$ and $t \models B$;
- $t \models \sim A$ iff $t \not\models A$;
- $t \models \sharp_i(A_1, \dots, A_{r_i})$ iff $\mathcal{A} \models \Psi_i(t, \hat{A}_1, \dots, \hat{A}_{r_i})$, where $\hat{A} = \{s \in S \mid s \models A\}$;
- $t \models (Q_i y) A(y)$ iff $\mathcal{A} \models \Psi'_i(t, \widehat{\lambda y A(y)})$, where $\widehat{\lambda y A(y)} = \{(t, y) \mid t \models A(y)\}$.

4. We say A holds in $\mathbf{m}$ iff $\mathcal{A} \models \Psi_0(d, \hat{A})$.

5. The interpretation $\mathcal{I}$ induces a translation $*$ of $\mathbf{L}$ into a two-sorted language $\mathbf{L}^*$ based on the two domains S (of the algebra $\mathcal{A}$) and $U = \bigcup_t V_t$ (of the predicates of $\mathbf{L}$) as follows:

- each atomic predicate $P(x_1, \dots, x_n)$ (interpreted over the domain U) is translated into

$$[P(x_1, \dots, x_n)]_t^* = P^*(t, x_1, \dots, x_n),$$

where P^* is a two-sorted predicate with one more variable t ranging over S and $x_1, \dots, x_n$ ranging over U ;

- $[A \wedge B]_t^* = [A]_t^* \wedge [B]_t^*$;
- $[\sim A]_t^* = \sim [A]_t^*$;
- $[\sharp_i(A_1, \dots, A_{r_i})]_t^* = \Psi_i(t, \lambda s[A_1]_s^*, \dots, \lambda s[A_{r_i}]_s^*)$;
- $[(Q_i y)A(y)]_t^* = \Psi'_i(t, \lambda y \lambda s[A(y)]_s^*)$;
- Let $\|A\|^* = \Psi_0(d, \lambda t[A]_t^*)$.

6. It is easy to show by induction that:

- $t \models A$ iff $[A]_t^*$ holds in the naturally defined two-sorted model.

The reader should compare this definition with [Gabbay, 1996, Definition 3.2.6] and with Chapter 5 of [Gabbay *et al.*, 1994].

Gabbay's book on LDS contains plenty of examples of such systems.

In the particular case of temporal logic, the algebra has the form $(D, <, d)$, where D is the flow of time and $<$ is the earlier–later relation. $d \in D$ is the present moment (actual world).

5.3 Sample temporal completeness proof

The previous section presented the LDS semantics. This section will choose a sample temporal logical system and present *LDS* proof rules and a completeness theorem for it. We choose the modal logic $\mathbf{K}_t$ (See Section 3.2 of [Gabbay *et al.*, 1994]). This is the propositional logic with H and G complete for all Kripke frames (S, R, a) , $a \in S$, such that R is transitive and irreflexive. A wff A is a theorem of $\mathbf{K}_t$ iff for all models (S, R, a, h) with assignment h , we have $a \models A$.

We want to turn $\mathbf{K}_t$ into a quantified logic $Q\mathbf{K}_t$. We take as semantics the class of all models of the form (S, R, a, V, h) such that V_t for $t \in S$ is the domain of world t . The following is assumed to hold:

- $tRs \wedge sRs'$ and $a \in V_{s'}$ and $a \in V_t$ imply $a \in V_s$ (i.e. elements are born, exist for a while and then possibly die).

DEFINITION 55 (Traditional semantics for QK_t).

1. A QK_t Kripke structure has the form (S, R, a, V, h) , where S is a non-empty set of possible worlds, $R \subseteq S^2$ is the irreflexive and transitive accessibility relation and $a \in S$ is the actual world. V is a function giving for each $t \in S$ a non-empty domain V_t .

- Let $V_S = \bigcup_{t \in S} V_t$.

h is the assignment function assigning for each t and each n -place atomic predicate P its extension $h(t, P) \subseteq V_t^n$, and for each constant c of the language its extension $h(c) \in V_S$.

Each n -place function symbol of the language of the form $f(x_1, \dots, x_n)$ and t is assigned a function $h(f) : V_S^n \mapsto V_S$.

Note that function symbols are rigid, i.e. the assignment to a constant c is a fixed rigid element which may or may not exist at a world t .²

2. Satisfaction $\models$ is defined in the traditional manner.

- (a) h can be extended to arbitrary terms by the inductive clause $h(f(x_1, \dots, x_n)) = h(f)(h(x_1), \dots, h(x_n))$.

- (b) We define for atomic P and terms $x_1, \dots, x_n$

$$t \models_h P(x_1, \dots, x_n) \text{ iff } (h(x_1), \dots, h(x_n)) \in h(t, P).$$

- (c) The cases of the classical connectives are the traditional ones.

- (d) $t \models_h \exists x A(x)$ iff for some $a \in V_t, t \models_h A(a)$.

- (e) $t \models_h \forall x A(x)$ iff for all $a \in V_t, t \models_h A(a)$.

3. $t \models_h GA(a_1, \dots, a_n)$ (resp. $t \models HA$) iff for all s , such that tRs (resp. sRt) and $a_1, \dots, a_n \in V_s, s \models A$.

4. $t \models U(A(a_1, \dots, a_n), B(b_1, \dots, b_k))$ iff for some s, tRs and $a_i \in V_s, i = 1, \dots, n$, we have $s \models A$ and for all s', tRs' and $s'Rt$ and $b_j \in V_{s'}, j = 1, \dots, k$, imply $s' \models B$. (The mirror image holds for $S(A, B)$.)

5. Satisfaction in the model is defined as satisfaction in the actual world.

²Had we wanted non-rigid semantics we would have stipulated that the extension of a function symbol is $h(t, f) : V_t^n \mapsto V_t$. There is no technical reason for this restriction and our methods still apply. We are just choosing a simpler case to show how LDS works. Note that by taking $h(t, P) \subseteq V_t^n$ as opposed to $h(t, P) \subseteq V_S^n$, we are introducing peculiarities in the semantic evaluation. $t \models P(a_1, \dots, a_n)$ becomes false if not all a_i are in V_t . We can insist that we give values to $t \models A(a_1, \dots, a_n)$ only if all elements are in V_t , but then what value do we give to $t \models GA$? One option is to let $t \models GA(a_1, \dots, a_n)$ iff for all s such that tRs and such that all $a_i \in V_s$ we have $s \models A$.

Anyway, there are many options here and a suitable system can probably be chosen for any application area.

Note that the logic $Q\mathbf{K}_t$ is not easy to axiomatize traditionally.

We now define an LDS corresponding to the system $Q\mathbf{K}_t$.

DEFINITION 56 (The algebra of labels).

1. Consider the first-order theory of one binary relation $<$ and a single constant d . Consider the axiom $\partial = \forall x \sim (x < x) \wedge \forall xyz(x < y \wedge y < z \rightarrow x < z)$. Any classical model of this theory has the form $\mathbf{m} = (S, R, a, g)$, where S is the domain, R is a binary relation on S giving the extension of the syntactical ' $<$ ' and g gives the extension of the variables and of d . $g(d)$ equals a and is the interpretation of the constant ' d '. Since $(S, R, a, g) \models \partial$, we have that R is irreflexive and transitive.
2. Let $U = \{t_1, t_2, \dots\}$ be a set of additional constants in the predicate language of $<$ and d . Let $\mathcal{A}$ be the set of all terms of the language. By a diagram $\Delta = (D, <, d)$, with $D = (D_1, D_2)$, we mean a set $D_1 \subseteq \mathcal{A}$, $d \in D_1$ of constants and variables and a set D_2 of formulae $\varphi(t, s)$ of the form $t < s, \sim (t < s), t = s, t \neq s$, with constants and variables from D_1 .
3. A structure $\mathbf{m} = (S, R, a, g)$ is a model of Δ iff the following hold:
 - (a) $g : D_1 \mapsto S$, with $g(d) = a$;
 - (b) R is irreflexive and transitive (i.e. it is a model of ∂);
 - (c) whenever $\varphi(t, s) \in D_2$ then $\varphi(g(t), g(s))$ holds in the model.
4. Note that g assigns elements of S also to the variables of D_1 . Let x be a constant or a variable. Denote by $g =_x g'$ iff for all variables and constants $y \neq x$ we have $g(y) = g'(y)$.

DEFINITION 57 (LDS language for $Q\mathbf{K}_t$).

1. Let $\mathbf{L}$ be the predicate modal language with the following:
 - (a) Its connectives and quantifiers are $\wedge, \vee, \rightarrow, \perp, \top, \forall, \exists, G, F, H, P$.
 - (b) Its variables are $\{x_1, x_2, \dots\}$.
 - (c) It has atomic predicates of different arities.
 - (d) It has function symbols $e_1, e_2, \dots$, of different arities.
 - (e) Let $\mathcal{A}$ be the language of the algebra of labels of Definition 56. We assume that $\mathcal{A}$ may share variables with $\mathbf{L}$ but its constants are distinct from the constants of $\mathbf{L}$. For each constant $t \in U$ of $\mathcal{A}$ and each natural number n , we assume we have in $\mathbf{L}$ a sequence of n -place function symbols

$$c_{n,1}^t(x_1, \dots, x_n), c_{n,2}^t(x_1, \dots, x_n) \dots$$

parameterized by t .

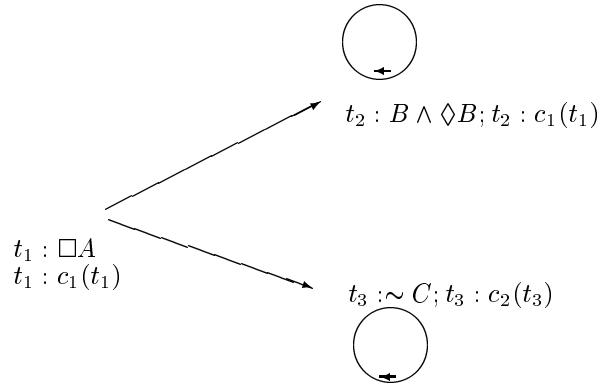


Figure 1.

Thus in essence we want an infinite number of Skolem functions of any arity parameterized by any $t \in U$. The elements of $\mathcal{A}$ are our labels.

The LDS language is presented as $(\mathcal{A}, \mathbf{L})$.

2. A declarative unit is a pair $t : A$, where t is a constant from $\mathcal{A}$ and A is a wff of $\mathbf{L}$.

Note that because some of the function symbols of $\mathbf{L}$ are parameterized from U , we can get labels in $\mathcal{A}$ as well. For example,

$$t : P(x, c_{1,1}^t(x))$$

is a declarative unit.

3. A configuration has the form $(D, <, \mathbf{f}, d, U)$, where $(D, <, d)$ is a diagram as defined in Definition 56 and $\mathbf{f}$ and U are functions associating with each $t \in D_1$ a set $\mathbf{f}(t)$ of wffs of $\mathbf{L}$ and a set U_t of terms of $\mathbf{L}$.

We also write:

$t : A$ to indicate that $A \in \mathbf{f}(t)$;

$t : c$ to indicate that $c \in U_t$.

Configurations can also be presented graphically. See for example Fig. 1.

DEFINITION 58 (LDS semantics for $Q\mathbf{K}_t$).

1. A model for an LDS language $(\mathcal{A}, \mathbf{L})$ has the form $\mathbf{n} = (S, R, a, V, h, g)$ where (S, R, a, V, h) is a traditional $Q\mathbf{K}_t$ model for the language $\mathbf{L}$ and g is an assignment from the labelling language $\mathcal{A}$ into S , giving values in S to each label and each variable. The following must hold:
 - (a) for a variable or a constant x common to both $\mathcal{A}$ and $\mathbf{L}$, $g(x) = h(x)$;
 - (b) for any $c = c_{n,k}^t(x_1, \dots, x_n)$ we have $h(x) \in V_{g(t)}$;
 - (c) let $t_1(x), t_2(y)$ be two terms of $\mathbf{L}$ containing the subterms x and y . Assume that ∂ (when augmented with the function symbols of $\mathbf{L}$ and equality axioms) satisfies

$$\partial \vdash_{\mathcal{A}} (x = y) \rightarrow (t_1 = t_2);$$
 then if $g(x) = g(y)$ then $h(t_1) = h(t_2)$.
 - (d) $g(d) = a$.
2. Let $\Delta = (D, <, \mathbf{f}, d, U)$ be a modal configuration in a language $\mathbf{L}$. We define the notion of $\mathbf{n} \models \Delta$ to mean that the following hold:
 - (a) for every $t \in D$ and $A \in \mathbf{f}(t)$ we have $g(t) \models_h A$ in (S, R, a, V, h) , according to Definition 55;
 - (b) $(S, R, a, g) \models (D, <, d)$ according to Definition 56;
 - (c) if $x \in D_1$ is a variable then for all $g' =_x g$ we have $\mathbf{n}' = (S, R, a, V, h, g') \models \Delta$, provided $\mathbf{n}'$ is an acceptable model (satisfying the restrictions in 1). This means that the free variables occurring in D are interpreted universally.
3. Let $\mathbf{n} = (S, R, a, V, h, g)$ be a model in a language $(\mathcal{A}, \mathbf{L})$. Let $(\mathcal{A}', \mathbf{L}')$ be an extension of the language. Then $\mathbf{n}'$ is said to be an extension of $\mathbf{n}$ to $(\mathcal{A}', \mathbf{L}')$ if the restriction of V', h', g' to $(\mathcal{A}, \mathbf{L})$ equals V, h, g respectively.
4. Let Δ be a temporal configuration in $(\mathcal{A}, \mathbf{L})$ and Δ' in $(\mathcal{A}', \mathbf{L}')$. We write $\Delta \models \Delta'$ iff any model $\mathbf{n}$ of Δ can be extended to a model $\mathbf{n}'$ of Δ' .
5. We write $\emptyset \models \Delta$ or equivalently $\models \Delta$ iff for any model $\mathbf{n}$, $\mathbf{n} \models \Delta$.

DEFINITION 59 (LDS proof rules for $Q\mathbf{K}_t$). Let $\Delta = (D, <, \mathbf{f}, d, U)$ and $\Delta' = (D', <', \mathbf{f}', d, U')$ be two temporal configurations. We say that Δ' is obtained from Δ by the application of a single forward proof rule if one of the following cases hold (let $\diamond \in \{P, F\}$, and $\square \in \{G, H\}$).

1. $\Diamond$ **introduction case**

For some $t, s \in D_1, t < s \in D_2, A \in \mathbf{f}(s)$ and Δ' is the same as Δ except that $\mathbf{f}'(t) = \mathbf{f}(t) \cup \{\Diamond A\}$.

Symbolically we can write $\Delta' = \Delta_{[t < s; s:A]}$ for F and $\Delta_{[s < t, s:A]}$ for P .

2. $\Diamond$ **elimination case**

For some $t \in D_1$ and some $A, \Diamond A \in \mathbf{f}(t)$ and for some new atomic $s \in U$, that does not appear in Δ , we have that Δ' is the same as Δ except that $D'_1 = D_1 \cup \{s\}$. $D'_2 = D_2 \cup \{t < s\}$ for $\Diamond = fF$ (resp. $s < t$ for $\Diamond = P$). $\mathbf{f}'(s) = \{A\}$.

Symbolically we can write $\Delta' = \Delta_{[t:\Diamond A;s]}$.

3. $\Box$ **elimination case**

For some $t, s \in D_1$ such that $t < s \in D_2$ for $\Box = G$ (resp. $s < t$ for $\Box = H$) we have $\Box A \in \mathbf{f}(t)$ and Δ' is like Δ except that $\mathbf{f}'(s) = \mathbf{f}(s) \cup \{A\}$.

Symbolically we can write $\Delta' = \Delta_{[t < s; t:GA]}$, and $\Delta' = \Delta_{[s < t; t:HA]}$.

4. **Local classical case**

Δ' is like Δ except that for $t \in D_1$ we have $\mathbf{f}'(t) = \mathbf{f}(t) \cup \{A\}$, where A follows from $\mathbf{f}(t)$ using classical logic inference only.³

Symbolically we can write $\Delta' = \Delta_{[t \vdash A]}$.

5. **Local $\forall$ elimination**

For some $t \in D_1$ we have $\forall x A(x) \in \mathbf{f}(t)$ and $c \in U_t$, Δ' is like Δ except $\mathbf{f}'(t) = \mathbf{f}(t) \cup \{A(c)\}$.

Symbolically we can write $\Delta' = \Delta_{[t:\forall x A(x); x/c]}$.

6. **Local $\exists$ elimination**

For some $t \in D_1$ and $\forall x_1, \dots, x_n \exists y A(x_1, \dots, x_n, y) \in \mathbf{f}(t)$ and some *new* function symbol $c^t(x_1, \dots, x_n)$ we have

$$\mathbf{f}'(t) = \mathbf{f}(t) \cup \{\forall x_1, \dots, x_n A(x_1, \dots, x_n, c^t(x_1, \dots, x_n))\}.$$

Otherwise Δ' is like Δ .

Symbolically we can write $\Delta' = \Delta_{[t:\forall x_1, \dots, x_n \exists y A; y/c^t]}$.

³Every formula of QK_t can be presented in the form $B(Q_1/\Box_1 A_1, \dots, Q_n/\Box_n A_n)$ where $\Box_i \in \{G, H\}$ and $B(Q_1, \dots, Q_n)$ is a modal free classical formula and A_i are general QK_t formulae. P/A means the substitution of A for P . A set of formulae of the form $B_i(Q_j^i/\Box_j A_j^i)$ proves using classical rules only a formula $B(Q_j/\Box_j A_j)$ iff $\{B_i(Q_j^i)\}$ proves classically $B(Q_j)$.

7. Visa rules

- (a) For $t, s \in D_1, t < s < t' \in D_2$ and $c \in U_t$ and $c \in U_{t'}$, Δ' is like Δ except that $U'_s = U_s \cup \{c\}$.
- (b) $U'_t = U_t \cup \{c^t\}$. In other words any c^t can be put in the domain of world t .

Symbolically we write $\Delta' = \Delta_{[t:c, t':c \text{ to } s:c]}$ and $\Delta' = \Delta_{[t:c^t]}$ respectively.

8. Inconsistency rules

- (a) If $t, s \in D_1$ and $\perp \in \mathbf{f}(t)$ then let Δ' be like Δ except that $\mathbf{f}'(s) = \mathbf{f}(s) \cup \{\perp\}$ and $\mathbf{f}'(x) = \mathbf{f}(x)$, for $x \neq s$.
Symbolically we write $\Delta' = \Delta_{[t:\perp \text{ to } s:\perp]}$.
- (b) If $(D, <, d)$ is classically inconsistent and $s \in D_1$, we let Δ' be as in (a) above.
Symbolically we write $\Delta' = \Delta_{[\perp \text{ to } s:\perp]}$.

9. $\Box$ introduction rule

We say Δ' is obtained from Δ by a single-level $n+1$ introduction rule if the following holds. For some $t \in D_1$ we have $\mathbf{f}'(t) = \mathbf{f}(t) \cup \{\Box A\}$, $\Box \in \{G, H\}$ and Δ_2 follows from Δ_1 using a sequence of applications of single, forward rules or of single-level $m \leq n$ introduction rules, and Δ_1 is like Δ except that

$$D_1^1 = D_1 \cup \{t_1\}, D_2^1 = D_2 \cup \{t < t_1\} \text{ for } \Box = G$$

$$(\text{and respectively } D_2^1 = D_2 \cup \{t_1 < t\} \text{ for } \Box = H)$$

where t_1 is a completely new constant label and $\mathbf{f}^1(t_1) = \{\top\}$.

Δ_2 is like Δ_1 except that $\mathbf{f}^2(t_1) = \mathbf{f}^1(t_1) \cup \{A\}$.

Symbolically we write $\Delta' = \Delta_{[t:\Box A]}$.

10. Diagram rule

Δ' is an extension of Δ as a diagram, i.e. $D \subseteq D', \mathbf{f}'; (t) = \mathbf{f}(t), t \in D_1$ and $\mathbf{f}'(t) = \{\top\}$, for $t \in D'_1 - D_1$, and we have $(D, <, d) \vdash_{\mathcal{A}} (D', <, d)$.

Symbolically we write $\Delta' = \Delta_{[D \vdash D']}$.

Note that for our logic, QK_t , this rule just closes $<$ under transitivity.

- 11. We write $\Delta \vdash \Delta'$ iff there exists a sequence of steps of any level leading from Δ to Δ' . We write $\vdash \Delta'$ if $\Delta_0 \vdash \Delta'$, for Δ_0 the theory with $\{d\}$ only with $\mathbf{f}(d) = \{\top\}$.

Let t be a label of Δ . We write $\Delta \vdash t : A$ iff for any Δ' such that $\Delta \vdash \Delta'$ there exists a Δ'' such that $\Delta' \vdash \Delta''$ and $A \in \mathbf{f}''(t)$. In other

words, Δ can be proof-theoretically manipulated until A is proved at node t .

12. Notice that if $\Delta \vdash \Delta'$ then the language $\mathbf{L}'$ of Δ' is slightly richer in labels and Skolem functions than the language $\mathbf{L}$.

Note also that if $\Delta \vdash \Delta'$ then there is a sequence of symbols $\pi_1, \dots, \pi_n$ such that $\Delta' = \Delta_{\pi_1, \dots, \pi_n}$.

In fact Δ' is uniquely determined up to symbolic isomorphism by the sequence $\pi_1, \dots, \pi_n$.

13. **Local cut rule**

$\vdash$ of item 11 above is without the following *local cut rule*. Let $\Delta_{t:B}$ denote the database obtained from Δ by adding B at label t . Then $\Delta_{t:B} \vdash \Delta'$ and $\Delta_{t:\sim B} \vdash \Delta'$ imply $\Delta \vdash \Delta'$.

14. The consequence $\Delta \vdash \Delta'$ can be implicitly formulated in a ‘sequent’-like form as follows:

- $\Delta \vdash \Delta_\pi$ (axiom)
for π as in any of items 1–10 above.
- $$\frac{\Delta \vdash \Delta'; \Delta' \vdash \Delta''}{\Delta \vdash \Delta''}.$$
- Local cut rule.

DEFINITION 60 (Inconsistency).

1. A theory $\Delta = (D, <, \mathbf{f}, d, U)$ is immediately inconsistent iff either $(D, <, d)$ is inconsistent as a classical diagram or for some $t \in D_1$, $\perp \in \mathbf{f}(t)$.
2. Δ is inconsistent iff $\Delta \vdash \Delta'$ and Δ' is immediately inconsistent.
3. Note that if we do not provide the following inconsistency rules, namely

$$\frac{t : \perp}{s : \perp}$$

and

$$\frac{(D, <, d) \text{ is inconsistent}}{s : \perp},$$

then two inconsistent modal configurations cannot necessarily prove each other.

THEOREM 61 (Soundness). *Let Δ be in $\mathbf{L}$ and Δ' in $\mathbf{L}'$. Then $\Delta \vdash \Delta'$ implies $\Delta \models \Delta'$. In words, if $\mathbf{n}$ is a model in $\mathbf{L}$ such that $\mathbf{n} \models \Delta$, then $\mathbf{n}$ can be extended to $\mathbf{n}' \models \Delta'$, where $\mathbf{n}'$ is like $\mathbf{n}$ except that the assignments are extended to $\mathbf{L}'$. In particular $\vdash \Delta'$ implies $\models \Delta'$.*

Proof. By induction on the number of single steps proving Δ' from Δ . We can assume Δ is consistent. It is easy enough to show that if Δ' is obtained from Δ by a single step then there is an $\mathbf{n}' \models \Delta'$.

Let $\mathbf{n} = (S, R, a, V, h, g)$ be a model of $\Delta = (D, <, \mathbf{f}, d, U)$. Let Δ' be obtained from Δ by a single proof step and assume $\mathbf{n} \models \Delta$. We show how to modify $\mathbf{n}$ to an $\mathbf{n}'$ such that $\mathbf{n}' \models \Delta'$.

We follow the proof steps case by case:

1. $\Diamond$ **introduction case**

Here take $\mathbf{n}' = \mathbf{n}$.

2. $\Diamond$ **elimination case**

Here Δ' contains a completely new constant $s \in D'_1$. We can assume g is not defined for this constant. Since $\mathbf{n} \models \Delta$, we have $g(t) \models \Diamond A$ and so for some $b \in S, g(t)Rb$ and $b \models A$. Let g' be like g except $g'(s) = b$. Then $\mathbf{n}' \models \Delta'$.

3. $\Box$ **elimination case**

Here let $\mathbf{n}' = \mathbf{n}$.

4. **Local classical case**

Let $\mathbf{n}' = \mathbf{n}$.

5. **Local $\forall$ elimination**

Let $\mathbf{n}' = \mathbf{n}$.

6. **Local $\exists$ elimination**

We can assume the new function $c^t(x_1, \dots, x_n)$ is new to the language of Δ and that h is not defined for c^t . Since in $\mathbf{n}$, $t \models \forall x_1, \dots, x_n \exists y A(x_1, \dots, x_n, y)$, for every $(x_1, \dots, x_n) \in V_t^n$ a $y \in V_t$ exists such that $t \models A(x_i, y)$. Thus an assignment $h'(c^t)$ can be given to c^t by defining it to be this y for $(x_1, \dots, x_n) \in V_t^n$ and to be a fixed element of V_t for tuples $(x_1, \dots, x_n)$ not in V_t^n . Let $\mathbf{n}'$ be like $\mathbf{n}$ except we take h' instead of h .

7. **Visa rules**

(a) Take $\mathbf{n}' = \mathbf{n}$.

(b) Take $\mathbf{n}' = \mathbf{n}$.

(c) Take $\mathbf{n}' = \mathbf{n}$.

8. Inconsistency rules are not applicable as we assume Δ is consistent.

9. Let $\mathbf{n}_1$ be a model for the language of Δ_1 and assume $\mathbf{n}_1 \models \Delta_1$. Then by the induction hypothesis there exists $\mathbf{n}_2 \models \Delta_2$, where $\mathbf{n}_2$ extends $\mathbf{n}_1$.

Assume now that $\mathbf{n} \not\models \Delta'$. Then $t \not\models \Box A$, and hence for some b such that $tRb, b \not\models A$.

Let $\mathbf{n}_1$ be defined by extending g to $g_1(t_1) = b$. Clearly $\mathbf{n}_1 \models \Delta_1$. Since $\Delta_1 \vdash \Delta_2$, g_1 can be extended to g_2 so that $\mathbf{n}_2 \models \Delta_2$, i.e. $b \models A$, but this contradicts our previous assumption. Hence $\mathbf{n} \models \Delta'$.

10. Let $\mathbf{n}' = \mathbf{n}$.

The above completes the proof of soundness. ■

THEOREM 62 (Completeness theorem for LDS proof rules). *Let Δ be a consistent configuration in $\mathbf{L}$; then Δ has a model $\mathbf{n} \models \Delta$.*

Proof. We can assume that there are an infinite number of constants, labels and variables in $\mathbf{L}$ which are not mentioned in Δ . We can further assume that there are an infinite number of Skolem functions $c_{n,i}^t(x_1, \dots, x_n)$ in $\mathbf{L}$, which are not in Δ , for each t and n .

Let $\delta(n)$ be a function such that

$$\delta(n) = (t_n, B_n, t'_n, \alpha_n, k_n),$$

where t_n, t'_n are labels or variables, α_n a term of $\mathbf{L}$ and B_n a formula of $\mathbf{L}$, and k_n a number between 0 and 7.

Assume that for each pair of labels t, t' of $\mathbf{L}$ and each term α and each formula B of $\mathbf{L}$ and each $0 \leq k \leq 7$ there exist an infinite number of numerals m such that $\delta(m) = (t, B, t', \alpha, k)$.

Let $\Delta_0 = \Delta$. Assume we have defined $\Delta_n = (D^n, <, \mathbf{f}^n, d, U^n)$ and it is consistent.

We now define Δ_{n+1} . Our assumption implies that $(D^n, <, d)$ is classically consistent as well as each $\mathbf{f}^n(t)$. Consider $(t_n, B_n, t'_n, \alpha_n, k_n)$. It is possible that the formulae and labels of $\delta(n)$ are not even in the language of Δ_n , but if they are we can carry out a construction by case analysis.

Case $k_n = 0$

This case deals with the attempt to add either the formula B_n or its negation $\sim B_n$ to the label t_n in D^n . We need to do that in order to build eventually a Henkin model. We first try to add B_n and see if the resulting database $(\Delta_n)_{t_n : B_n}$ is LDS-consistent. If not then we try to add $\sim B_n$. If both are LDS-inconsistent then Δ_n itself is LDS-inconsistent, by the local cut rule (see item 13 of Definition 59).

The above is perfectly acceptable if we are prepared to adopt the local cut rule. However, if we want a system without cut then we must try to add B_n or $\sim B_n$ using possibly other considerations, maybe a different notion of local consistency, which may be heavily dependent on our particular logic.

The kind of notion we use will most likely be correlated to the kind of traditional cut elimination available to us in that logic.

Let us motivate the particular notion we use for our logic, while at the same time paying attention to the principles involved and what possible variations are needed for slightly different logics.

Let us consider an LDS-consistent theory $\Delta = (D, <, \mathbf{f}, d, U)$ and an arbitrary B . We want a process which will allow us to add either B or $\sim B$ at node t (i.e. form $\Delta_{t:B}$ or $\Delta_{t:\sim B}$) and make sure that the resulting theory is LDS-consistent. We do not have the cut rule and so we must use a notion of local consistency particularly devised for our particular logic.

Let us try the simplest notion, that of classical logic consistency. We check whether $\mathbf{f}(t) \cup \{B\}$ is classically consistent. If yes, take $\Delta_{t:B}$, otherwise take $\Delta_{t:\sim B}$. However, the fact that we have classical consistency at t does not necessarily imply that the database is LDS-consistent. Consider $\{t : FA \rightarrow \sim B, s : A; t < s\}$. Here $\mathbf{f}(t) = FA \rightarrow \sim B$. $\mathbf{f}(s) = A$. If we add B to t then we still have local consistency but as soon as we apply the F introduction rule at t we get inconsistency.

Suppose we try to be more sophisticated. Suppose we let $\mathbf{f}_*(t) = \mathbf{f}(t) \cup \{F^k X \mid X \in \mathbf{f}(s), \text{ for some } k, t <^k s\} \cup \{Y \mid G^k Y \in \mathbf{f}(s), \text{ for some } k, s <^k t\}$ and try to add either B or $\sim B$ to $\mathbf{f}_*(t)$ and check for classical consistency. If neither is consistent then for some X_i, Y_k, X'_i, Y'_j we have

$$\begin{aligned} \mathbf{f}(t), X_i, Y_j &\vdash \sim B \\ \mathbf{f}(t), X'_i, Y'_j &\vdash B. \end{aligned}$$

Hence $\mathbf{f}_*(t)$ is inconsistent. However, we do have LDS rules that can bring the X and Y into $\mathbf{f}(t)$ which will make Δ LDS-inconsistent.

So at least one of B and $\sim B$ can be added. Suppose $\Delta_{t:B}$ is locally consistent. Does that make it LDS-consistent?

Well, not necessarily. Consider

$$\{t < s; t : F^k A \rightarrow \sim B; s : A\}$$

and assume we have a condition ∂_1 on the diagrams

$$\partial_1 = \forall xy(x < y \rightarrow \exists z(x < z < y)).$$

We would have to apply the diagram rule k times at the appropriate labels to get inconsistency.

It is obvious from the above discussion that to add B or $\sim B$ at node t we have to make it consistent with all possible wffs that can be proved using LDS rules to have label t (i.e. to be at t).

Let us define then, for t in Δ ,

$$\Delta_{(t)} = \{A \mid \Delta \vdash t : A\}.$$

We say B is locally consistent with Δ at t , or that $\mathbf{f}(t) \cup \{B\}$ is locally consistent iff it is classically consistent with $\Delta_{(t)}$. Note that if $\Delta_{(t)} \vdash A$ classically then $\Delta \vdash t : A$.

We can now proceed with the construction:

1. if $t_n \notin D_1^n$ then $\Delta_{n+1} = \Delta_n$;
2. if $t_n \in D_1^n$ and $\mathbf{f}^n(t_n) \cup \{B_n\}$ is locally consistent, then let $\mathbf{f}^{n+1}(t_n) = \mathbf{f}^n(t_n) \cup \{B_n\}$.
Otherwise $\mathbf{f}^n(t_n)$ is locally consistent with $\sim B_n$ and we let $\mathbf{f}^{n+1}(t_n) = \mathbf{f}^n(t_n) \cup \{\sim B_n\}$. For other $x \in D_1^n$, let $\mathbf{f}^{n+1}(x) = \mathbf{f}^n(x)$. Let $\Delta_{n+1} = (D^n, <, d, \mathbf{f}^{n+1}, U^n)$.

We must show the following:

- if Δ is LDS-consistent and $\Delta_{t_i:B_i}$ is obtained from Δ by simultaneously adding the wffs B_i to $\mathbf{f}(t_i)$ of Δ and if for all i $\mathbf{f}(t_i) \cup \{B_i\}$ is *locally*-consistent, then $\Delta_{t_i:B_i}$ is LDS-consistent.

To show this, assume that $\Delta_{t_i:B_i}$ is LDS-inconsistent. We show by induction on the complexity of the inconsistency proof that Δ is also inconsistent.

Case one step

In this case $\Delta_{t_i:B_i}$ is immediately inconsistent. It is clear that Δ is also inconsistent, because the immediate inconsistency cannot be at any label t_i , and the diagram is consistent.

Case $(l + 1)$ steps

Consider the proof of inconsistency of $\Delta_{t_i:B_i}$. Let π be the first proof step leading to this inconsistency. Let $\Delta' = (\Delta_{t_i:B_i})_\pi$. π can be one of several cases as listed in Definition 59. If π does not touch the labels t_i , then it can commute with the insertion of B_i , i.e. $\Delta' = (\Delta_\pi)_{t_i:B_i}$ and by the induction hypothesis Δ_π is LDS-inconsistent and hence so is Δ .

If π does affect some label t_i , we have to make a case analysis:

- $\pi = [t_i < s; s : A]$, i.e. $\Diamond A$ is put in t . In this case consider $(\Delta_\pi)_{t_i:B_i}$. If adding B_i is still locally consistent then by the induction hypothesis $(\Delta_\pi)_{t_i:B_i}$ is consistent. But this is Δ' and so Δ' is consistent. If adding B_i is locally inconsistent, this means that $\Delta_{(t_i)} \cup \{B_i\}$ classically proves $\perp$, contrary to assumption.
- $\pi = [s < t_i; s : GA]$ (resp. $\pi = [t_i < s; s : HA]$) i.e. A is put in t_i . The reasoning is similar to the previous case.
- $\pi = [t_i : \Box A]$, $\Box \in \{G, H\}$, the reasoning is similar to previous cases.
- $\pi = [t_i \vdash A]$, similar to the previous ones.
- $\pi =$ classical quantifier rules. This case is also similar to previous ones;

- $\pi = \text{visa rule}$. This means some new constants are involved in the new inconsistency from $\Delta_{(t_i)} \cup \{B_i\}$. These will turn into universal quantifiers and contradict the assumptions.
- The inconsistency rules are not a problem.
- The diagram rule does not affect t_i .
- π applies to B_i itself. Assume that $B_i = \Box C_i$ where $\Box$ is G (resp. H) and that C_i is put in some s , $t_i < s$ (resp. $s < t_i$).

Let us first check whether C_i is locally consistent at s . This will not be the case if $\Delta_{(s)} \vdash \sim C_i$. This would imply $\Delta_{(t)} \vdash \Diamond \sim C_i$ contradicting the fact that B_i is consistent with $\Delta_{(t)}$. Thus consider $\Delta_{t_i:B_i;s:C_i}$. This is the same as Δ' . It is *LDS*-inconsistent, by a shorter proof; hence by the induction hypothesis Δ is *LDS*-consistent;

- B_i is $\Diamond C_i$ and π eliminates $\Diamond$, i.e. a new point s is introduced with $t_i < s$ for $\Diamond = F$ (resp. $s < t_i$ for $\Diamond = P$) and $s : C_i$ is added to the database.

We claim that C_i is locally consistent in s , in Δ^+ , where Δ^+ is the result of adding s to Δ but not adding $s : C_i$. Otherwise $\Delta_{(s)}^+ \vdash \sim C_i$, and since s is a completely new constant and $\Delta^+ \vdash s : \sim C_i$ this means that $\Delta \vdash t_i : \Box \sim C_i$, a contradiction. Hence C_i is locally consistent in $\Delta_{(s)}^+$. Hence by the induction hypothesis if $\Delta_{s:C_i;t_i:B_i}^+$ is *LDS*-inconsistent so is Δ^+ . If Δ^+ is *LDS*-inconsistent then $\Delta^+ \vdash s : \sim C_i$ and hence $\Delta \vdash t : \Box \sim C_i$ contradicting the local consistency of B_i at t .

- π applies to B_i and π adds $\Diamond B_i$ to $s < t_i$ for $\Diamond = F$ (resp. $t_i < s$ for $\Diamond = P$).

Again we claim $\Diamond B_i$ is locally consistent at s . Otherwise $\Delta_{(s)} \vdash \Box \sim B_i$ and so B_i would not be locally consistent at t_i . We now consider $\Delta_{t_i:B_i;s:\Diamond B_i}$ and get a contradiction as before.

- π is a use of a local classical rule, i.e. $\mathbf{f}(t) \cup \{B_i\} \vdash C_i$, and π adds C_i at t_i .

We claim we can add $B_i \wedge C_i$ at t , because it is locally consistent. Otherwise $\Delta_{(t)} \vdash B_i \rightarrow \sim C_i$ contradicting consistency of $\Delta_{(t)} \cup \{B_i\}$ since $\mathbf{f}(t) \vdash B_i \rightarrow C_i$.

- π is a Skolemization on B_i or an instantiation from B_i . All these classical operations are treated as in the previous case.

Case $k_n = 1$

1. If $t_n, t'_n \in D_1^n$ and $t_n < t'_n \in D_2^n$ and $B_n \in \mathbf{f}^n(t'_n)$ then let $\mathbf{f}^{n+1}(t_n) = \mathbf{f}^n(t_n) \cup \{\Diamond B_n\}$ and $\mathbf{f}^{n+1}(x) = \mathbf{f}^n(x)$, for $x \neq t_n$. Let $\Delta_{n+1} = (D^n, <, \mathbf{f}^{n+1}, d, U^n)$.
2. Otherwise let $\Delta_{n+1} = \Delta_n$.

Case $k_n = 2$

1. If $t_n \in D_1^n$ and $B_n = \Diamond C \in \mathbf{f}^n(t_n)$, then let s be a completely new constant and let $D_1^{n+1} = D_1^n \cup \{s\}$, $D_2^{n+1} = D_2^n \cup \{t_n < s\}$ for $\Diamond = F$ (resp. $s < t_n$ for $\Diamond = P$). Let $\mathbf{f}^{n+1}$ be like $\mathbf{f}^n$ or D_1 and let $\mathbf{f}^{n+1}(s) = \{C\}$. Let $\Delta_{n+1} = (D^{n+1}, <, \mathbf{f}^{n+1}, d, U^n)$ and let the new domain at s, U_s^{n+1} , contain all free variables of C .
2. Otherwise let $\Delta_{n+1} = \Delta_n$.

Case $k_n = 3$

1. If $t_n, t'_n \in D_1^n$ and $t_n < t'_n \in D_2^n$ and $B_n = \Box C$ and $\Box = G$ (resp. $t'_n < t_n$ and $\Box = H$) and $B_n \in \mathbf{f}^n(t_n)$ and all free variables of C are in the domain $U_{t'_n}^n$ then let $\mathbf{f}^{n+1} = \mathbf{f}^n(x)$, for $x \neq t'_n$ and $\mathbf{f}^{n+1}(t'_n) = \mathbf{f}^n(x) \cup \{C\}$.
Let $\Delta_{n+1} = (D^n, <, \mathbf{f}^{n+1}, d, U^n)$.
2. Otherwise let $\Delta_{n+1} = \Delta_n$.

Case $k_n = 4$

1. We have $t_n \in D_1^n$ and $\mathbf{f}^n(t_n) \vdash B_n$ classically. Let $\mathbf{f}^{n+1}(x) = \mathbf{f}^n(x)$ for $x \neq t_n$ and let $\mathbf{f}^{n+1}(t_n) = \mathbf{f}^n(t_n) \cup \{B_n\}$.
Let $\Delta_{n+1} = (D^n, <, \mathbf{f}^{n+1}, d, U^n)$.
2. Otherwise let $\Delta_{n+1} = \Delta_n$.

Case $k_n = 5$

1. $t_n \in D_1^n$ and $B_n = \forall x C(x) \in \mathbf{f}^n(t_n)$ and $\alpha_n \in U^n$. Then let $\mathbf{f}^{n+1}(t_n) = \mathbf{f}^n(t_n) \cup \{C(\alpha_n)\}$ and $\mathbf{f}^{n+1}(x) = \mathbf{f}^n(x)$ for $x \neq t_n$ and $\Delta_{n+1} = (D^n, <, \mathbf{f}^{n+1}, d, U^n)$.
2. Otherwise let $\Delta_{n+1} = \Delta_n$.

Case $k_n = 6$

1. $t_n \in D_1^n$ and $B_n \in \mathbf{f}^n(t_n)$ and $B_n = \exists u C(u, y_1, \dots, y_k)$.

Let $c^{t_n}(y_1, \dots, y_k)$ be a completely new Skolem function of this arity not appearing in Δ_n and let

$$\mathbf{f}^{n+1}(t_n) = \mathbf{f}^n(t_n) \cup \{C(c^{t_n}(y_1, \dots, y_k), y_1, \dots, y_k)\}.$$

This is consistent by classical logic. Let $U_{t_n}^{n+1} = U_{t_n}^n \cup \{c^{t_n}(y_1, \dots, y_k)\}$.

Let U^{n+1} and $\mathbf{f}^{n+1}$ be the same as U^n and $\mathbf{f}^n$ respectively, for $x \neq t_n$. Take $\Delta_{n+1} = (D^n, <, \mathbf{f}^{n+1}, d, U^{n+1})$.

2. Otherwise let $\Delta_{n+1} = \Delta_n$.

Case $k_n = 7$

1. If $t_n, t'_n < s_n \in D_1^n$ and $t_n < t'_n < s_n \in D_2^n$ and $\alpha_n \in U_{t_n}^n \cap U_{s_n}^n$ then let $U_{t'_n}^{n+1} = U_{t'_n}^n \cup \{\alpha_n\}$ and $U_x^{n+1} = U_x^n$ for $x \neq t'_n$. Let $\Delta_{n+1} = (D^n, <, \mathbf{f}^n, d, U^{n+1})$.

2. Otherwise let $\Delta_{n+1} = \Delta_n$.

Let Δ_∞ be defined by $D_i^\infty = \bigcup_n D_i^n$, $\mathbf{f}^\infty = \bigcup_n \mathbf{f}^n$, $U^\infty = \bigcup_n U^n$.

Δ_∞ is our Henkin model. Let $\mathbf{n} = (S, R, a, V, h, g)$, where

$$S = D_1^\infty$$

$$R = \{(x, y) \mid x < y \in D_2^\infty\}$$

$$a = d$$

$$V_t = U_t$$

$$V = U^\infty$$

$$g = \text{identity};$$

then $h(t, P) = \{(x_1, \dots, x_n) \mid P(x_1, \dots, x_n) \in \mathbf{f}^\infty(t)\}$ for $t \in S$ and P n -place atomic predicate, and $x_1, \dots, x_n \in V_t$. $\blacksquare$

LEMMA 63.

1. $\mathbf{n}$ is an acceptable structure of the semantics.
2. For any t and B , $t \models B$ iff $B \in \mathbf{f}^\infty(t)$.

Proof.

1. We need to show that R is irreflexive and transitive. This follows from the construction and the diagram rule.

2. We prove this by induction. Assume $\Diamond A \in \mathbf{f}^\infty(t)$. Then for some n we have $t \in D_1^n$ and $\Diamond A \in \mathbf{f}^n(t)$. Thus at some $n' \geq n$ we put $s \in D_1^{n'}, t < s \in D_2^{n'}$ for $\Diamond = F$ (resp. $s < t$ for $\Diamond = P$) and $A \in \mathbf{f}^{n'}(s)$.

Assume $\Diamond A \notin \mathbf{f}^\infty(t)$. At some n , $B_n = \Diamond A$ and had $\mathbf{f}^n(t) \cup \{\Diamond A\}$ been consistent, B_n would have been put in $\mathbf{f}^{n+1}$.

Hence $\Box \sim A \in \mathbf{f}^{n+1}$.

Assume $t < s$ (resp. $s < t$), $s \in D_1^\infty$. Hence for some $n'' \geq n'$ we have $\delta(n'') = (t, \Box \sim A, s, -, 3)$. At this stage $\sim A$ would be in $\mathbf{f}^{n''+1}(s)$. Thus for all $s \in S$ such that tRs (resp. sRt) we have $s \models \sim A$.

The classical cases follow the usual Henkin proof.

This completes the proof of the lemma and the proof of Theorem 62. ■

5.4 Label-dependent connectives

We saw earlier that *since* and *until* cannot be defined from $\{G, H, F, P\}$ but if we allow names for worlds we can write

$$\sim q \wedge Gq \wedge Hq \rightarrow [U(A, B) \leftrightarrow F(A \wedge H(P \sim q \rightarrow B))].$$

We can introduce the label-dependent connectives $G^x A, H^x A$ meaning

$$t \models G^x A \text{ iff for all } y (t < y < x \rightarrow y \models A);$$

$$t \models H^x A \text{ iff for all } y (x < y < t \rightarrow y \models A).$$

We can then define

$$t : U(A, B) \text{ as } t : F(A \wedge H^t B).$$

Let $F^x A$ be $\sim G^x \sim A$ and $P^x A$ be $\sim H^x \sim A$. Then

$$t \models F^x A \text{ iff for some } y, t < y < x \text{ and } y \models A \text{ hold.}$$

$$t \models P^x A \text{ iff for some } y, x < y < t \text{ and } y \models A \text{ hold.}$$

Consider $t \models G^x \perp$ and $t \models F^x \top$. The first holds iff $\sim \exists y (t < y < x)$ which holds if either $\sim (t < x)$ or x is an immediate successor of t . The second holds if $\exists y (t < y < x)$.

Label-dependent connectives are very intuitive semantically since they just restrict the temporal range of the connectives. There are many applications where such connectives are used. In the context of LDS such connectives are also syntactically natural and offer no additional complexity costs.

We have the option of defining two logical systems. One is an ordinary predicate temporal logic (which is not an LDS) where the connectives G, H are labelled. We call it LQK_t (next definition). This is the logic analogous

to $Q\mathbf{K}_t$. The other system is an LDS formulation of $LQ\mathbf{K}_t$. This system will have (if we insist on being pedantic) two lots of labels: labels of the LDS and labels for the connectives. Thus we can write $t : F^x A$, where t is an LDS label from the labelling algebra $\mathcal{A}$ and x is a label from L ; when we give semantics, both t and x will get assigned possible worlds. So to simplify the LDS version we can assume $L = \mathcal{A}$.

DEFINITION 64 (The logic $LQ\mathbf{K}_t$).

1. Let L be a set of labels, and for each $x \in L$, let G^x and H^x be temporal connectives. The language of $LQ\mathbf{K}_t$ has the classical connectives, the traditional connectives G, H and the labelled connectives G^x, H^x , for each $x \in L$.
2. An $LQ\mathbf{K}_t$ model has the form (S, R, a, V, h, g) , where (S, R, A, V, h) is a $Q\mathbf{K}_t$ model (see Definition 55) and $g : L \mapsto S$, assigning a world to each label. Satisfaction for G^x (resp. H^x) is defined by

$$(3x) \quad t \models_{h,g} G^x(a_1, \dots, a_n) \text{ iff for all } s \text{ such that } tRs \wedge sRg(x) \text{ such that } a_1, \dots, a_n \in B_s, \text{ we have } s \models_{h,g} A.$$

The mirror image condition is required for H^x .

DEFINITION 65 (LDS version of $LQ\mathbf{K}_t$). Our definition is in parallel to Definitions 56–58. We have the added feature that the language $\mathbf{L}$ of the LDS allows for the additional connectives F^t, P^t, G^t, H^t , where t is from the labelling algebra. For this reason we must modify the LDS notion of an $LQ\mathbf{K}_t$ theory and require that all the labels appearing in the connectives of the formulae of the theory are also members of D_1 , the diagram of labels of the theory.

DEFINITION 66 (LDS proof rules for $LQ\mathbf{K}_t$). We modify Definition 59 as follows:

1. F^x introduction case
For some $t, s \in D_1, t < s < x \in D_2$ and $A \in \mathbf{f}(s)$, Δ' is the same as Δ except that $\mathbf{f}'(t) = \mathbf{f}(t) \cup \{F^x A\}$.

Symbolically we write

$$\Delta' = \Delta_{[t < s < x; s : A]}^x.$$

2. F^x elimination case
For some $t \in D_1$ and some $A, F^x A \in \mathbf{f}(t)$, and for some new atomic

$s \in U$ that does not appear in Δ , we have that Δ' is the same as Δ except that

$$\begin{aligned} D'_1 &= \Delta_1 \cup \{s\}. \\ D'_2 &= D_2 \cup \{t < s < x\}. \\ \mathbf{f}'(s) &= \{A\}. \end{aligned}$$

Note that it may be that $\Delta_2 \cup \{t < s < x\}$ is inconsistent in which case Δ is inconsistent.

3. G^x elimination case

For some $t \in D_1$ such that $t < s < x \in D_2$ we have $G^x A \in \mathbf{f}(t)$ and Δ' is like Δ except that $\mathbf{f}'(s) = \mathbf{f}(s) \cup \{A\}$.

The P^x, H^x rules are the mirror images of all the above and all the other rules remain the same.

4. G^x, H^x introduction case

This case is the same as in Definition 59 except that in the text we replace

$$D_2^1 = D_2 \cup \{t < t_1\}$$

by $D_2^1 = D_2 \cup \{t < t_1 < x\}$ for the case of G^x and the mirror image for the case of H^x .

Similarly we write $\Delta' = \Delta_{[t:\Box^x A]}$.

THEOREM 67 (Soundness and completeness). *The LDS version of LQK_t is sound and complete for the proposed semantics.*

Proof. The soundness and completeness are proved along similar lines to the QK_t case see Theorems 61 and 62. ■

6 TEMPORAL LOGIC PROGRAMMING

We can distinguish two views of logic, the declarative and the imperative. The declarative view is the traditional one, and it manifests itself both syntactically and semantically. Syntactically a logical system is taken as being characterized by its set of theorems. It is not important how these theorems are generated. Two different algorithmic systems generating the same set of theorems are considered as producing the same logic. Semantically a logic is considered as a set of formulae valid in all models. The model $\mathcal{M}$ is a static

semantic object. We evaluate a formula φ in a model and, if the result of the evaluation is positive (notation $\mathcal{M} \models \varphi$), the formula is valid. Thus the logic obtained is the set of all valid formulae in some class $\mathcal{K}$ of models.

In contrast to the above, the imperative view regards a logic syntactically as a dynamically generated set of theorems. Different generating systems may be considered as different logics. The way the theorems are generated is an integral part of the logic. From the semantic viewpoint, a logical formula is not *evaluated* in a model but performs *actions* on a model to get a *new* model. Formulae are accepted as valid according to what they do to models. For example, we may take φ to be valid in $\mathcal{M}$ if $\varphi(\mathcal{M}) = \mathcal{M}$. (i.e. $\mathcal{M}$ is a fixed point of φ).

Applications of logic in computer science have mainly concentrated on the exploitation of its declarative features. Logic is taken as a language for describing properties of models. The formula φ is evaluated in a model $\mathcal{M}$. If φ holds in $\mathcal{M}$ (evaluation successful) then $\mathcal{M}$ has property φ . This view of logic is, for example, most suitably and most successfully exploited in the areas of databases and in program specification and verification. One can present the database as a deductive logical theory and query it using logical formulae. The logical evaluation process corresponds to the computational querying process. In program verification, for example, one can describe in logic the properties of the programs to be studied. The description plays the role of a model $\mathcal{M}$. One can now describe one's specification as a logical formula φ , and the query whether φ holds in $\mathcal{M}$ (denoted $\mathcal{M} \vdash \varphi$) amounts to verifying that the program satisfies the specification. These methodologies rely solely on the declarative nature of logic.

Logic programming as a discipline is also declarative. In fact it advertises itself as such. It is most successful in areas where the declarative component is dominant, e.g. in deductive databases. Its procedural features are not imperative (in our sense) but computational. In the course of evaluating whether $\mathcal{M} \vdash \varphi$, a procedural reading of $\mathcal{M}$ and φ is used. φ does not imperatively act on $\mathcal{M}$, the declarative logical features are used to guide a procedure—that of taking steps for finding whether φ is true. What does not happen is that $\mathcal{M}$ and φ are read imperatively, resulting in some action. In logic programming such actions (e.g. `assert`) are obtained by side-effects and special non-logical imperative predicates and are considered undesirable. There is certainly no conceptual framework within logic programming for allowing only those actions which have logical meaning.

Some researchers have come close to touching upon the imperative reading of logic. Belnap and Green [1994] and the later so-called data semantics school regard a formula φ as generating an action on a model $\mathcal{M}$, and changing it. See [van Benthem, 1996]. In logic programming and deductive databases the handling of integrity constraints borders on the use of logic imperatively. Integrity constraints have to be maintained. Thus one can either reject an update or do some corrections. Maintaining integrity

constraints is a form of executing logic, but it is logically *ad hoc* and has to do with the local problem at hand. Truth maintenance is another form. In fact, under a suitable interpretation, one may view any resolution mechanism as model building which is a form of execution. In temporal logic, model construction can be interpreted as execution. Generating the model, i.e. finding the truth values of the atomic predicates in the various moments of time, can be taken as a sequence of execution.

As the need for the imperative executable features of logic is widespread in computer science, it is not surprising that various researchers have touched upon it in the course of their activity. However, there has been no conceptual methodological recognition of the imperative paradigm in the community, nor has there been a systematic attempt to develop and bring this paradigm forward as a new and powerful logical approach in computing.

The area where the need for the imperative approach is most obvious and pressing is temporal logic. In general terms, a temporal model can be viewed as a progression of ordinary models. The ordinary models are what is true at each moment of time. The imperative view of logic on the other hand also involves step-by-step progression in virtual ‘time’, involving both the syntactic generation of theorems and the semantic actions of a temporal formula on the temporal model. Can the two intuitive progressions, the semantic time and the action (transaction) time, be taken as the same? In the case of temporal logic the answer is ‘yes’. We can act upon the models in the same time order as their chronological time. This means acting on earlier models first. In fact intuitively a future logical statement can be read (as we shall see) both declaratively and imperatively. Declaratively it describes what should be true, and imperatively it describes the actions to be taken to ensure that it becomes true. Since the chronology of the action sequence and the model sequence are the same, we can virtually *create* the future model by our actions. The logic USF, presented in Chapter 10 of [Gabbay *et al.*, 1994], was the first attempt at promoting the imperative view as a methodology, with a proposal for its use as a language for controlling processes.

The purpose of this section is twofold:

1. to present a practical, sensible, logic programming machine for handling time and modality;
2. to present a general framework for extending logic programming to non-classical logics.

Point 1 is the main task of this section. It is done within the framework of 2.

Horn clause logic programming has been generalized in essentially two major ways:

1. using the metalevel features of ordinary Horn clause logic to handle time while keeping the syntactical language essentially the same;
2. enriching the syntax of the language with new symbols and introducing additional computation rules for the new symbols.

The first method is basically a simulation. We use the expressive power of ordinary Horn clause logic to talk about the new features. The *Demo* predicate, the *Hold* predicate and other metapredicates play a significant role.

The second method is more direct. The additional computational rules of the second method can be broadly divided into two:

2.1 Rewrites

2.2 Subcomputations

The rewrites have to do with simplifying the new syntax according to some rules (basically eliminating the new symbols and reducing goals and data to the old Horn clause language) and the subcomputations are the new computations which arise from the reductions.

Given a temporal set of data, this set has the intuitive form:

‘ $A(x)$ is true at time t ’.

This can be represented in essentially two ways (in parallel to the two methods discussed):

1. adding a new argument for time to the predicate A , writing $A^*(t, x)$ and working within an ordinary Horn clause computational environment;
2. leaving time as an external indicator and writing ‘ $t : A(x)$ ’ to represent the above temporal statement.

To compare the two approaches, imagine that we want to say the following:

‘If $A(x)$ is true at t , then it will continue to be true’.

The first approach will write it as

$$\forall s (A^*(t, x) \wedge t < s \rightarrow A^*(s, x)).$$

The second approach has to talk about t . It would use a special temporal connective ‘ G ’ for ‘always in the future’. Thus the data item becomes

$$t : A(x) \rightarrow GA(x).$$

It is equivalent to the following in the first approach:

$$A^*(t, x) \rightarrow \forall s[t < s \rightarrow A^*(s, x)].$$

The statement ‘ GA is true at t ’ is translated as

$$\forall s(t < s \rightarrow A^*(s)).$$

The second part of this section introduces temporal connectives and wants to discover what kind of temporal clauses for the new temporal language arise in ordinary Horn clause logic when we allow time variables in the atoms (e.g. $A^*(t, x), B^*(s, y)$) and allow time relations like $t < s, t = s$ for time points. This would give us a clue as to what kind of temporal Horn clauses to allow in the second approach. The computational tractability of the new language is assured, as it arises from Horn clause computation. Skolem functions have to be added to the Horn clause language, to eliminate the F and P connectives which are existential. All we have to do is change the computational rules to rely on the more intuitive syntactical structure of the second approach.

6.1 Temporal Horn clauses

Our problem for this section is to start with Horn clause logic with the ability to talk about time through time coordinates, and see what expressive power in term of connectives (P, F, G, H , etc.) is needed to do the same job. We then extend Prolog with the ability to compute directly with these connectives. The final step is to show that the new computation defined for P, F, G, H is really ordinary Prolog computation modified to avoid Skolemization.

Consider now a Horn clause written in predicate logic. Its general form is of course $\bigwedge \text{atoms} \rightarrow \text{atom}$. If our atomic sentences have the form $A(x)$ or $R(x, y)$ or $Q(x, y)$ then these are the atoms one can use in constructing the Horn clause. Let us extend our language to talk about time by following the first approach; that is, we can add time points, and allow special variables t, s to range over a flow of time $(T, <)$ (T can be the set of integers, for example) and write $Q^*(t, x, y)$ instead of $Q(x, y)$, where $Q^*(t, x, y)$ (also written as $Q(t, x, y)$, abusing notation) can be read as

$$‘Q(x, y) \text{ is true at time } t.’$$

We allow the use of $t < s$ to mean ‘ t is earlier than s ’. Recall that we do not allow mixed atomic sentences like $x < t$ or $x < y$ or $A(t, s, x)$ because these would read as

‘John loves 1980 at 1979’ or

‘John < 1980’ or

‘John < Mary’.

Assume that we have organized our Horn clauses in such a manner: what kind of time expressive power do we have? Notice that our expressive power is potentially increased. We are committed, when we write a formula of the form $A(t, x)$, to t ranging over time and x over our domain of elements. Thus our model theory for classical logic (or Horn clause logic) does not accept any model for $A(t, x)$, but only models in which $A(t, x)$ is interpreted in this very special way. Meanwhile let us examine the syntactical expressive power we get when we allow for this two-sorted system and see how it compares with ordinary temporal and modal logics, with the connectives P, F, G, H .

When we introduce time variables t, s and the earlier-later relation into the Horn clause language we are allowing ourselves to write more atoms. These can be of the form

$$A(t, x, y) \\ t < s$$

(as we mentioned earlier, $A(t, s, y), x < y, t < y, y < t$ are excluded).

When we put these new atomic new sentences into a Horn clause we get the following possible structures for Horn clauses. $A(t, x), B(s, y)$ may also be *truth*.

$$(a0) \quad A(t, x) \wedge B(s, y) \rightarrow R(u, z). \\ \text{Here } < \text{ is not used.}$$

$$(a1) \quad A(t, x) \wedge B(s, y) \wedge t < s \rightarrow R(u, z). \\ \text{Here } t < s \text{ is used in the body but the time variable } u \text{ is not the same} \\ \text{as } t, s \text{ in the body.}$$

$$(a2) \quad A(t, x) \wedge B(s, y) \wedge t < s \rightarrow R(t, z). \\ \text{Same as (a1) except the time variable } u \text{ appears in the body as } u = t.$$

$$(a3) \quad A(t, x) \wedge B(s, y) \wedge t < s \rightarrow R(s, z). \\ \text{Same as (a1) with } u = s.$$

$$(a4) \quad A(t, x) \wedge B(s, y) \rightarrow R(t, z). \\ \text{Same as (a0) with } u = t, \text{ i.e. the variable in the head appears in the} \\ \text{body.}$$

The other two forms (b) and (c) are obtained when the head is different: (b) for time independence and (c) for a pure $<$ relation.

$$(b) \quad A(t, x) \wedge B(s, y) \rightarrow R1(z)$$

$$(b') \quad A(t, x) \wedge B(s, y) \wedge t < s \rightarrow R1(z).$$

$$(c) \quad A(t, x) \wedge B(s, y) \rightarrow t < s.$$

$$(d) \quad A((1970, x), \text{ where } 1970 \text{ is a } \textit{constant} \text{ date.}$$

Let us see how ordinary temporal logic with additional connectives can express directly, using the temporal connectives, the logical meaning of the above sentences. Note that if time is linear we can assume that one of $t < s$ or $t = s$ or $s < t$ always occur in the body of clauses because for linear time

$$\vdash \forall t \forall s [t < s \vee t = s \vee s < t],$$

and hence $A(t, x) \wedge B(s, y)$ is equivalent to

$$(A(t, x) \wedge B(t, y)) \vee (A(t, x) \wedge B(s, y) \wedge t < s) \vee (A(t, x) \wedge B(s, y) \wedge s < t).$$

Ordinary temporal logic over linear time allows the following connectives:

Fq , read: ‘ q will be true’

Pq , read: ‘ q was true’

$$\Diamond q = q \vee Fq \vee Pq$$

$$\Box q = \sim \Diamond \sim q$$

$\Box q$ is read: ‘ q is always true’.

If $[A](t)$ denotes, in symbols, the statement that A is true at time t , then we have

$$[Fq](t) \equiv \exists s > t ([q](s))$$

$$[Pq](t) \equiv \exists s < t ([q](s))$$

$$[\Diamond q](t) \equiv \exists s ([q](s))$$

$$[\Box q](t) \equiv \forall s ([q](s)).$$

Let us see now how to translate into temporal logic the Horn clause sentences mentioned above.

Case (a0)

Statement (a0) reads

$$\forall t \forall s \forall u [A(t, x) \wedge B(s, y) \rightarrow R(u, z)].$$

If we push the quantifiers inside we get

$$\exists t A(t, x) \wedge \exists s B(s, y) \rightarrow \forall u R(u, z),$$

which can be written in the temporal logic as

$$\Diamond A(x) \wedge \Diamond B(y) \rightarrow \Box R(z).$$

If we do not push the $\forall u$ quantifier inside we get $\Box(\Diamond A(x) \wedge \Diamond B(y) \rightarrow R(z))$.

Case (a1)

The statement (a1) can be similarly seen to read (we do not push $\forall t$ inside)

$$\forall t\{A(t, x) \wedge \exists s[B(s, y) \wedge t < s] \rightarrow \forall uR(u, z)\}$$

which can be translated as: $\Box\{A(x) \wedge FB(y) \rightarrow \Box R(z)\}$. Had we pushed $\forall t$ to the antecedent we would have got

$$\exists t[A(t, x) \wedge \exists s(B(s, y) \wedge t < s)] \rightarrow \forall uR(u, z),$$

which translates into

$$\Diamond[A(x) \wedge FB(y)] \rightarrow \Box R(z).$$

Case (a2)

The statement (a2) can be rewritten as

$$\forall t[A(t, x) \wedge \exists s(B(s, y) \wedge t < s) \rightarrow R(t, z)],$$

and hence it translates to $\Box(A(x) \wedge FB(y) \rightarrow R(z))$.

Case (a3)

Statement (a3) is similar to (a2). In this case we push the external $\forall t$ quantifier in and get

$$\forall s[\exists t[A(t, x) \wedge t < s] \wedge B(s, y) \rightarrow R(s, z)],$$

which translates to

$$\Box[PA(x) \wedge B(y) \rightarrow R(z)].$$

Case (a4)

Statement (a4) is equivalent to

$$\forall t[A(t, x) \wedge \exists sB(s, y) \rightarrow R(t, z)],$$

and it translates to

$$\Box(A(x) \wedge \Diamond B(y) \rightarrow R(z)).$$

Case (b)

The statement (b) is translated as

$$\Diamond(A(x) \wedge \Diamond B(y)) \rightarrow R1(z).$$

Case (b')

The statement (b') translates into

$$\Diamond(A(x) \wedge FB(y)) \rightarrow R1(z).$$

Case (c)

Statement (c) is a problem. It reads $\forall t \forall s [A(t, x) \wedge B(s, y) \rightarrow t < s]$; we do not have direct connectives (without negation) to express it. It says for any two moments of time t and s if $A(x)$ is true at t and $B(y)$ true at s then $t < s$. If time is linear then $t < s \vee t = s \vee s < t$ is true and we can write the conjunction

$$\sim \Diamond(A(x) \wedge PB(y)) \wedge \Diamond(A(x) \wedge B(y)).$$

Without the linearity of time how do we express the fact that t *should be* '<-related to s '?

We certainly have to go beyond the connectives $P, F, \Diamond, \Box$ that we have allowed here.

Case (d)

$A(1970, x)$ involves a constant, naming the date 1970. The temporal logic will also need a propositional constant *1970*, which is true *exactly* when the time is 1970, i.e.

$$\mathbf{M}t \models 1970 \text{ iff } t = 1970.$$

Thus (d) will be translated as $\Box(1970 \rightarrow A(x))$. *1970* can be read as the proposition 'The time now is 1970'.

The above examples show what temporal expressions we can get by using Horn clauses with time variables as an *object* language. We are not discussing here the possibility of 'simulating' temporal logic in Horn clauses by using the Horn clause as a *metalanguage*. Horn clause logic can do that to any logic as can be seen from Hodges [Hodges, 1985].

DEFINITION 68. The language contains $\wedge, \rightarrow, F$ (it will be the case) P (it was the case) and $\Box$ (it is always the case).

We define the notions of:

Ordinary clauses;

Always clauses;

Heads;

Bodies;

Goals.

1. A *clause* is either an always clause or an ordinary clause.
2. An *always clause* is $\Box A$ where A is an ordinary clause.
3. An *ordinary clause* is a head or an $A \rightarrow H$ where A is a body and H is a head.
4. A *head* is either an atomic formula or FA or PA , where A is a conjunction of ordinary clauses.
5. A *body* is an atomic formula, a conjunction of bodies, an FA or a PA , where A is a body.
6. A goal is any body.

EXAMPLE 69.

$$a \rightarrow F((b \rightarrow Pq) \wedge F(a \rightarrow Fb))$$

is an acceptable clause.

$$a \rightarrow \Box b$$

is not an acceptable clause.

The reason for not allowing $\Box$ in the head is computational and not conceptual. The difference between a (temporal) logic programming machine and a (temporal) automated theorem prover is *tractability*. Allowing disjunctions in heads or $\Box$ in heads crosses the boundary of tractability. We can give computational rules for richer languages and we will in fact do so in later sections, but we will lose tractability; what we will have then is a theorem prover for full temporal logic.

EXAMPLES 70.

$$\begin{aligned} &P[F(FA(x) \wedge PB(y) \wedge A(y)) \wedge A(y) \wedge B(x)] \rightarrow \\ &P[F(A(x) \rightarrow FP(Q(z) \rightarrow A(y)))] \end{aligned}$$

is an ordinary clause. So is $a \rightarrow F(b \rightarrow Pq) \wedge F(a \rightarrow Fb)$, but not $A \rightarrow \Box b$.

First let us check the expressive power of this *temporal Prolog*. Consider

$$a \rightarrow F(b \rightarrow Pq).$$

This is an acceptable clause. Its predicate logic reading is

$$\forall t[a(t) \rightarrow \exists s > t[b(s) \rightarrow \exists u < sq(u)]].$$

Clearly it is more directly expressive than the Horn clause *Prolog* with time variables. Ordinary Prolog can rewrite the above as

$$\forall t(a(t) \rightarrow \exists s(t < s \wedge (b(s) \rightarrow \exists u(u < s \wedge q(u))))) ,$$

which is equivalent to

$$\forall t(\exists s\exists u(a(t) \rightarrow t < s \wedge (b(s) \rightarrow u < s \wedge q(u))).$$

If we Skolemize with $\mathbf{s}_0(t)$ and $\mathbf{u}_0(t)$ we get the clauses

$$\forall t[(a(t) \rightarrow t < \mathbf{s}_0(t)) \wedge (a(t) \wedge b(\mathbf{s}_0(t)) \rightarrow \mathbf{u}_0(t) < \mathbf{s}_0(t)) \wedge \\ (a(t) \wedge b(\mathbf{s}_0(t)) \rightarrow q(\mathbf{u}_0(t)))].$$

The following are representations of some of the problematic examples mentioned in the previous section.

$$(a1) \quad \Box(A(x) \wedge FB(y) \rightarrow \Box R(z)).$$

This is not an acceptable always clause but it can be equivalently written as

$$\Box(\Diamond(A(x) \wedge FB(y)) \rightarrow R(z)).$$

$$(a2) \quad \Box(A(x) \wedge FB(y) \rightarrow R(z)).$$

$$(b') \quad \Diamond(A(x) \wedge FB(y)) \rightarrow R1(z).$$

(b) can be written as the conjunction below using the equation

$$\begin{aligned} \Diamond q &= Fq \vee Pq \vee q : \\ (A(x) \wedge FB(y) &\rightarrow R1(z)) \wedge \\ (F(A(x) \wedge FB(y)) &\rightarrow R1(z)). \end{aligned}$$

(a2) Can be similarly written.

(c) can be written as

$$\forall t, s(A(x)(t) \wedge B(y)(s) \rightarrow t < s).$$

This is more difficult to translate. We need negation as failure here and write

$$\begin{aligned} \Box(A(x) \wedge PB(y) &\rightarrow \perp) \\ \Box(A(x) \wedge B(y) &\rightarrow \perp) \end{aligned}$$

From now on we continue to develop the temporal logic programming machine.

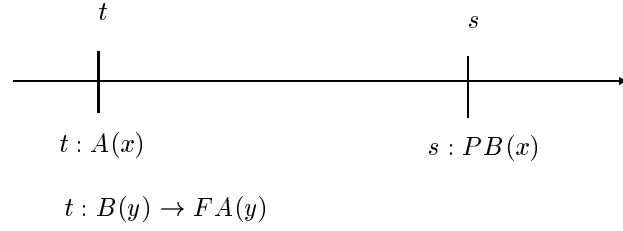


Figure 2. A temporal configuration.

6.2 LDS—Labelled Deductive System

This section will use the labelled deductive methodology of the previous section as a framework for developing the temporal Prolog machine. We begin by asking ourselves what is a temporal database? Intuitively, looking at existing real temporal problems, we can say that we have information about things happening at different times and some connections between them. Figure 2 is such an example.

The diagram shows a finite set of points of time and some labelled formulae which are supposed to hold at the times indicated by the labels. Notice that we have labelled not only assertions but also Horn clauses showing dependences across times. Thus at time t it may be true that B will be true. We represent that as $t : FB$. The language we are using has F and P as connectives. It is possible to have more connectives and still remain within the Horn clause framework. Most useful among them are ' $t : F^s A$ ' and ' $t : P^s A$ ', reading ' $t < s$ and $s : A$ ' and ' $s < t$ and $s : A$ '. In words: ' A will be true at time $s > t$ '.

The temporal configuration comprises two components.

1. A (finite) graph $(\rho, <)$ of time points and the temporal relationships between them.
2. With each point of the graph we associate a (finite) set of clauses and assertions, representing what is true at that point.

In Horn clause computational logic, there is an agreement that if a formula of the form $A(x) \rightarrow B(x)$ appears in the database with x free then it is understood that x is universally quantified. Thus we assume $\forall x(A(x) \rightarrow B(x))$ is in the database. The variable x is then called universal (or type 1). In the case of modal and temporal logics, we need another type of variable, called type 2 or a Skolem variable. To explain the reason, consider the item of data

$$'t : FB(x)'$$

This reads, according to our agreement,

$$' \forall x FB(x) \text{ true at } t.'$$

For example, it might be the sentence: t : 'Everyone will leave'.

The time in the future in which $B(x)$ is true depends on x . In our example, the time of leaving depends on the person x . Thus, for a given unknown (uninstantiated) u , i.e. for a given person u which we are not yet specifying, we know there must be a point t_1 of time (t_1 is dependent on u) with $t_1 : B(u)$. This is the time in which u leaves.

This u is by agreement not a type 1 variable. It is a u to be chosen later. Really u is a Skolem constant and we do not want to and cannot read it as $t_1 : \forall u B(u)$. Thus we need two types of variables. The other alternative is to make the dependency of t_1 on u explicit and to write

$$t_1(x) : B(x)$$

with x a universal type 1 variable, but then the object language variable x appears in the world indices as well. The world indices, i.e. the t , are external to the formal clausal temporal language, and it is simpler not to mix the t and the x . We chose the two types of variable approach. Notice that when we ask for a goal $?G(u)$, u is a variable to be instantiated, i.e. a type 2 variable. So we have these variables anyway, and we prefer to develop a systematic way of dealing with them.

To explain the role of the two types of variables, consider the following classical Horn clause database and query:

$$\begin{aligned} A(x, y) \rightarrow B(x, y) \quad ?B(u, u) \\ A(a, a). \end{aligned}$$

This means 'Find an instantiation u_0 of u such that $\forall x, y [A(x, y) \rightarrow B(x, y)] \wedge A(a, a) \vdash B(u_0, u_0)$ '. There is no reason why we cannot allow for the following

$$\begin{aligned} A(u, y) \rightarrow B(x, u) \quad ?B(u, u) \\ A(a, a). \end{aligned}$$

In this case we want to find a u_0 such that

$$\forall x, y [A(u_0, y) \rightarrow B(x, u_0)] \wedge A(a, a) \vdash B(u_0, u_0)$$

or to show

$$\vdash \exists u \{ [\forall x, y [A(u, y) \rightarrow B(x, u)] \wedge A(a, a) \rightarrow B(u, u) \}$$

u is called a type 2 (Skolem) variable and x, y are universal type 1 variables. Given a database and a query of the form $\Delta(x, y, u)?Q(u)$, success means $\vdash \exists u[\forall x, y \Delta(x, y, u) \rightarrow Q(u)]$.

The next sequence of definitions will develop the syntax of the temporal Prolog machine. A lot depends on the flow of time. We will give a general definition (Definition 73 below), which includes the following connectives:

$\square$ Always.

F It will be the case.

P It was the case.

G It will always be the case (not including now).

H It has always been the case (up to now and not including now);.

$\bigcirc$ Next moment of time (in particular it implies that such a moment of time exists).

$\bullet$ Previous moment of time (in particular it implies that such a moment of time exists).

Later on we will also deal with S (Since) and U (Until).

The flows of time involved are mainly three:

- general partial orders $(T, <)$;
- linear orders;
- the integers or the natural numbers.

The logic and theorem provers involved, even for the same connectives, are different for different partial orders. Thus the reader should be careful to note in which flow of time we are operating. Usually the connectives $\bigcirc$ and $\bullet$ assume we are working in the flow of time of integers.

Having fixed a flow of time $(T, <)$, the temporal machine will generate finite configurations of points of time according to the information available to it. These are denoted by $(\rho, <)$. We are supposed to have $\rho \subseteq T$ (more precisely, ρ will be homomorphic into T), and the ordering on ρ will be the same as the ordering on T . The situation gets slightly complicated if we have a new point s and we do not know where it is supposed to be in relation to known points. We will need to consider all possibilities. Which possibilities do arise depend on $(T, <)$, the background flow of time we are working with. Again we should watch for variations in the sequel.

DEFINITION 71. Let $(\rho, <)$ be a finite partial order. Let $t \in \rho$ and let s be a new point. Let $\rho' = \rho \cup \{s\}$, and let $<'$ be a partial order on ρ' . Then

$(\rho', <, t)$ is said to be a (one new point) *future* (resp. *past*) *configuration* of $(\rho, <, t)$ iff $t <' s$ (resp. $s <' t$) and $\forall xy \in \rho(x < y \leftrightarrow x <' y)$.

EXAMPLE 72. Consider a general partial flow $(T, <)$ and consider the subflow $(\rho, <)$.

The possible future configurations (relative to $T, <$) of one additional point s are displayed in Fig. 3.

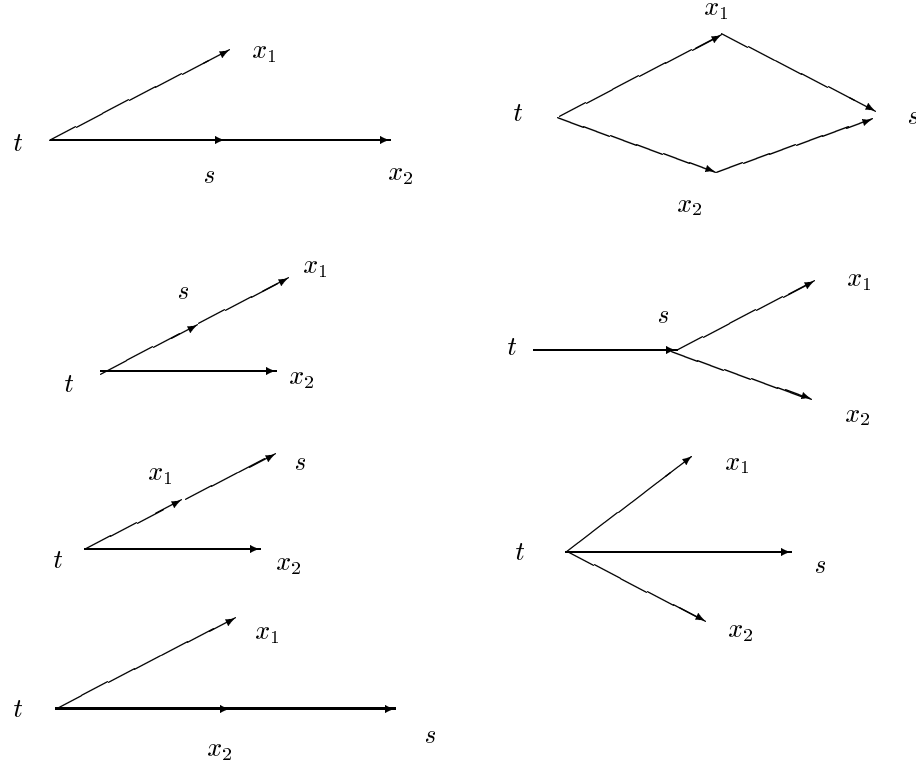


Figure 3.

For a finite $(\rho, <)$ there is a finite number of future and past non-isomorphic configurations. This finite number is exponential in the size of ρ . So in the general case without simplifying assumptions we will have an intractable exponential computation. A configuration gives all possibilities of putting a point in the future or past.

In the case of an ordering in which a next element or a previous element exists (like $t + 1$ and $t - 1$ in the integers) the possibilities for configurations

are different. In this case we must assume that we know the *exact* distance between the elements of $(\rho, <)$.

For example, in the configuration $\{t < x_1, t < x_2\}$ of Fig. 4 we may have the further following information as part of the configuration:

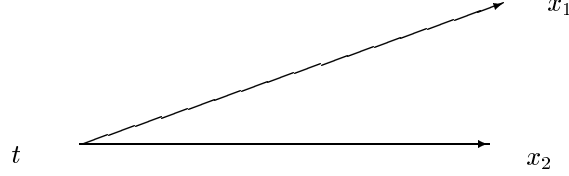


Figure 4.

$$t = \bullet^{18} x_1$$

$$t = \bullet^6 x_2$$

so that we have only a finite number of possibilities for putting s in.

Note that although $\bullet$ operates on propositions, it can also be used to operate on points of time, denoting the predecessor function.

DEFINITION 73. Consider a temporal Prolog language with the following connectives and predicates:

1. atomic predicates;
2. function symbols and constants;
3. two types of variables:
 universal variables (type 1) $V = \{x_1, y_1, z_1, x_2, y_2, z_2, \dots\}$
 and Skolem variables (type 2) $U = \{u_1, v_1, u_2, v_2, \dots\}$;
4. the connectives $\wedge, \rightarrow, \vee, F, P, \bigcirc, \bullet, \square$ and $\neg$.

FA reads: it will be the case that A .

PA reads: it was the case that A .

$\bigcirc A$ reads: A is true tomorrow (if a tomorrow exists; if tomorrow does not exist then it is false).

$\bullet A$ reads: A was true yesterday (if yesterday does not exist then it is false).

$\neg$: represents negation by failure.

We define now the notions of an *ordinary clause*, an *always clause*, a *body*, a *head* and a *goal*.

1. A *clause* is either an always clause or an ordinary clause.
2. An *always clause* has the form $\Box A$, where A is an ordinary clause.
3. An *ordinary clause* is a head or an $A \rightarrow H$, where A is a body and H is a head.
4. A *head* is an atomic formula or an FA or a PA or an $\bigcirc A$ or an $\bullet A$, where A is a finite conjunction of ordinary clauses.
5. A *body* is an atomic formula or an FA or a PA or an $\bigcirc A$ or an $\bullet A$ or $\neg A$ or a conjunction of bodies where A is a body.
6. A *goal* is a body whose variables are *all* Skolem variables.
7. A disjunction of goals is also a goal.

REMARK 74. Definition 73 included all possible temporal connectives. In practice different systems may contain only some of these connectives. For example, a modal system may contain only $\Diamond$ (corresponding to F) and $\Box$. A future discrete system may contain only $\bigcirc$ and F etc.

Depending on the system and the flow of time, the dependences between the connectives change. For example, we have the equivalence

$$\Box(a \rightarrow \bigcirc b) \text{ and } \Box(\bullet a \rightarrow b)$$

whenever both $\bullet a$ and $\bigcirc b$ are meaningful.

DEFINITION 75. Let $(T, <)$ be a flow of time. Let $(\rho, <)$ be a finite partial order. A *labelled temporal database* is a set of labelled ordinary clauses of the form $(t_i : A_i)$, $t \in \rho$, and always clauses of the form $\Box A_i$, A_i a clause. A labelled goal has the form $t : G$, where G is a goal.

Δ is said to be a labelled temporal database over $(T, <)$ if $(\rho, <)$ is homomorphic into $(T, <)$.

DEFINITION 76. We now define the computation procedure for the temporal Prolog for the language of Definitions 73 and 75. We assume a flow of time $(T, <)$. $\rho \subseteq T$ is the finite set of points of time involved so far in the computation. The exact computation steps depend on the flow of time. It is different for branching, discrete linear, etc. We will give the definition for linear time, though not necessarily discrete. Thus the meaning of $\bigcirc A$ in this logic is that there exists a next moment and A is true at this next moment. Similarly for $\bullet A$. $\bullet A$ reads: there exists a previous moment and A was true at that previous moment.

We define the success predicate $\mathbf{S}(\rho, <, \Delta, G, t, G_0, t_0, \Theta)$ where $t \in \rho$, $(\rho, <)$ is a finite partial order and Δ is a set of labelled clauses $(t : A)$, $t \in \rho$.

$\mathbf{S}(\rho, <, \Delta, G, t, G_0, t_0, \Theta)$ reads: the labelled goal $t : G$ succeeds from Δ under the substitution Θ to all the type 2 variables of G and Δ in the computation with starting labelled goal $t_0 : G_0$.

When Θ is known, we write $\mathbf{S}(\rho, <, \Delta, G, t, G_0, t_0)$ only.

We define the simultaneous success and failure of a set Π of metapredicates of the form $\mathbf{S}(\rho, <, \Delta, G, t, G_0, t_0)$ under a substitution Θ to type 2 variables. To explain the intuitive meaning of success or failure, assume first that Θ is a substitution which grounds all the Skolem type 2 variables. In this case (Π, Θ) succeeds if by definition all $\mathbf{S}(\rho, <, \Delta, G, t, G_0, t_0, \Theta) \in \Pi$ succeed and (Π, Θ) fails if at least one of $\mathbf{S} \in \Pi$ fails. The success or failure of $\mathbf{S}$ for a Θ as above has to be defined recursively. For a general Θ , (Π, Θ) succeeds, if for some Θ' such that $\Theta\Theta'$ grounds all type 2 variables $(\Pi, \Theta\Theta')$ succeeds. (Π, Θ) fails if for all Θ' such that $\Theta\Theta'$ grounds all type 2 variables we have that $(\Pi, \Theta\Theta')$ fails. We need to give recursive procedures for the computation of the success and failure of (Π, Θ) . In the case of the recursion, a given (Π, Θ) will be changed to a (Π', Θ') by taking $\mathbf{S}(\rho, <, \Delta, G, t, G_0, t_0) \in \Pi$ and replacing it by $\mathbf{S}(\rho', <', \Delta', G', t', G_0, t_0)$. We will have several such changes and thus get several Π' by replacing several $\mathbf{S}$ in Π . We write the several possibilities as (Π'_i, Θ'_i) . If we write (Π, Θ) to mean (Π, Θ) succeeds and $\sim (\Pi, \Theta)$ to read (Π, Θ) fails, then our recursive computation rules have the form: (Π, Θ) succeeds (or fails) if some Boolean combination of (Π'_i, Θ'_i) succeeds (or fails). The rules allow us to pick an element in Π , e.g. $\mathbf{S}(\rho, <, \Delta, G, t, G_0, t_0)$, and replace it with one or more elements to obtain the different (Π'_i, Θ'_i) , where Θ'_i is obtained from Θ . In case of failure we require that Θ grounds all type 2 variables. We do not define failure for a non-grounding Θ .

To summarize the general structure of the rules is:

(Π, Θ) succeeds (or fails) if some Boolean combination of the successes and failures of some (Π'_i, Θ'_i) holds and (Π, Θ) and (Π'_i, Θ'_i) are related according to one of the following cases:

Case I If $\Pi = \emptyset$ then (Π, Θ) succeeds (i.e. the Boolean combination of (Π_i, Θ_i) is *truth*).

Case II (Π, Θ) fails if for some $\mathbf{S}(\rho, <, \Delta, G, t, G_0, t_0)$ in Π we have G is atomic and for all $\Box(A \rightarrow H) \in \Delta$ and for all $(t : A \rightarrow H) \in \Delta, H\Theta$ does *not* unify with $G\Theta$. Further, for all Ω and s such that $t = \Omega s$ and for all $s : A \rightarrow \Omega H$ and all $\Box(A \rightarrow \Omega H)$ we have $H\Theta$ does not unify with $G\Theta$, where Ω is a sequence of $\bigcirc$ and $\bullet$.

REMARK 77. We must qualify the conditions of the notion of failure. If we have a goal $t : G$, with G atomic, we know for sure that $t : G$ finitely fails under a substitution Θ , if $G\Theta$ cannot unify with any head of a clause. This is what the condition above says. What are the candidates for unification? These are either clauses of the form $t : A \rightarrow H$, with H atomic, or $\Box(A \rightarrow H)$, with H atomic.

Do we have to consider the case where H is not atomic? The answer depends on the flow of time and on the configuration $(\rho, <)$ we are dealing

with. If we have, say, $t : A \rightarrow FG$ then if $A \rightarrow FG$ is true at t , G would be true (if at all) in some s , $t < s$. This s is irrelevant to our query $?t : G$. Even if we have $t' < t$ and $t' : A \rightarrow FG$ and A true at t' , we can still ignore this clause because we are not assured that any s such that $t' < s$ and G true at s would be the desired t (i.e. $t = s$).

The only case we have to worry about is when the flow of time and the configuration are such that we have, for example, $t' : A \rightarrow \bigcirc^5 G$ and $t = \bigcirc^5 t'$.

In this case we must add the following clause to the notion of failure: for every s such that $t = \bigcirc^n s$ and every $s : A \rightarrow \bigcirc^n H$, $G\Theta$ and $H\Theta$ do not unify.

We also have to check what happens in the case of always clauses.

Consider an integer flow of time and the clause $\Box(A \rightarrow \bigcirc^5 \bullet^{27} H)$. This is true at the point $s = \bullet^5 \bigcirc^{27} t$ and hence for failure we need that $G\Theta$ does not unify with $H\Theta$.

The above explains the additional condition on failure.

The following conditions 1–10, 12–13 relate to the success of (Π, Θ) if (Π'_i, Θ'_i) succeed. Condition (11) uses the notion of failure to give the success of negation by failure. Conditions 1–10, 12–13 give certain alternatives for success. They give failure if each one of these alternatives ends up in failure.

1. Success rule for atomic query:

$\mathbf{S}(\rho, <, \Delta, G, t, G_0, t_0) \in \Pi$ and G is atomic and for some head H , $(t : H) \in \Delta$ and for some substitutions Θ_1 to the universal variables of H and Θ_2 to the existential variables of H and G we have $H\Theta_1\Theta\Theta_2 = G\Theta\Theta_2$ and $\Pi' = \Pi - \{\mathbf{S}(\rho, <, \Delta, G, t, G_0, t_0)\}$ and $\Theta' = \Theta\Theta_2$.

2. Computation rule for atomic query:

$\mathbf{S}(\rho, <, \Delta, G, t, G_0, t_0) \in \Pi$ and G is atomic and for some $(t : A \rightarrow H) \in \Delta$ or for some $\Box(A \rightarrow H) \in \Delta$ and for some Θ_1, Θ_2 , we have $H\Theta_1\Theta\Theta_2 = G\Theta\Theta_2$ and $\Pi' = (\Pi - \{\mathbf{S}(\rho, <, \Delta, G, t, G_0, t_0)\}) \cup \{\mathbf{S}(\rho, <, \Delta, A\Theta_1, t, G_0, t_0)\}$ and $\Theta' = \Theta\Theta_2$.

The above rules deal with the atomic case. Rules 3, 4 and 4* deal with the case the goal is FG . The meaning of 3, 4 and 4* is the following. We ask FG at t . How can we be sure that FG is true at t ? There are two possibilities, (a) and (b):

- (a) We have $t < s$ and at $s : G$ succeeds. This is rule 3;
- (b) Assume that we have the fact that $A \rightarrow FB$ is true at t . We ask for A and succeed and hence FB is true at t . Thus there should exist a point s' in the future of t where B is true. Where can s' be? We don't know where s' is in the future of t . So we consider all future

configurations for s' . This gives us all future possibilities where s' can be. We assume for each of these possibilities that B is true at s' and check whether either G follows at s' or FG follows at s' . If we find that for all future constellations of where s' can be $G \vee FG$ succeeds in s' from B , then FG holds at t . Here we use the transitivity of $<$. Rule 4a gives the possibilities where s' is an old point s in the future of t ; Rule 4b gives the possibilities where s' is a new point forming a new configuration. Success is needed from *all* possibilities.

3. Immediate rule for F :

$\mathbf{S}(\rho, <, \Delta, FG, t, G_0, t_0) \in \mathbf{\Pi}$ and for some $s \in \rho$ such that $t < s$ we have $\mathbf{\Pi}' = (\mathbf{\Pi} - \{\mathbf{S}(\rho, <, \Delta, FG, t, G_0, t_0)\}) \cup \{\mathbf{S}(\rho, <, \Delta, G, s, G_0, t_0)\}$ and $\Theta' = \Theta$.

4. First configuration rule for F :

$\mathbf{S}(\rho, <, \Delta, FG, t, G_0, t_0) \in \mathbf{\Pi}$ and for some $(t : A \rightarrow F \wedge_j B_j) \in \Delta$ and some Θ_1, Θ_2 we have both (a) and (b) below are true. A may not appear in which case we pretend $A = \text{truth}$.

- (a) For *all* $s \in \rho$ such that $t < s$ we have that
 $\mathbf{\Pi}'_s = (\mathbf{\Pi} - \{\mathbf{S}(\rho, <, \Delta, FG, t, G_0, t_0)\}) \cup \{\mathbf{S}(\rho, <, \Delta, E\Theta_1, t, G_0, t_0)\} \cup \{\mathbf{S}(\rho, <, \Delta \cup \{(s : B_j\Theta_1) \mid j = 1, 2, \dots\}, D, s, G_0, t_0)\}$
succeeds with $\Theta'_s = \Theta\Theta_2$ and $D = G \vee FG$ and $E = A$.
- (b) For all future configurations of $(\rho, <, t)$ with a new letter s , denoted by the form $(\rho_s, <_s)$, we have that
 $\mathbf{\Pi}'_s = (\mathbf{\Pi} - \{\mathbf{S}(\rho, <, \Delta, FG, t, G_0, t_0)\}) \cup \{\mathbf{S}(\rho, <, \Delta, E\Theta_1, t, G_0, t_0)\} \cup \{\mathbf{S}(\rho_s, <_s, \Delta \cup \{(s : B_j) \mid j = 1, 2, \dots\}, D, s, G_0, t_0)\}$
succeeds with $\Theta'_s = \Theta\Theta_2$ and $D = G \vee FG$ and $E = A$.

The reader should note that conditions 3, 4a and 4b are needed only when the flow of time has some special properties. To explain by example, assume we have the configuration of Fig. 5 and $\Delta = \{t : A \rightarrow FB, t' : C\}$ as data, and our query is $?t : FG$.

Then according to rules 3, 4 we have to check and succeed in all the following cases:

1. from rule 3 we check $\{t' : C, t : A \rightarrow FB\}?t' : G$;
2. from rule 4a we check $\{t' : C, t : A \rightarrow FB, t' : B\}?t' : G$;
3. from rule 4b we check $\{t' : C, t : A \rightarrow FB, s : B\}?s : G$;

for the three configurations of Fig. 6.

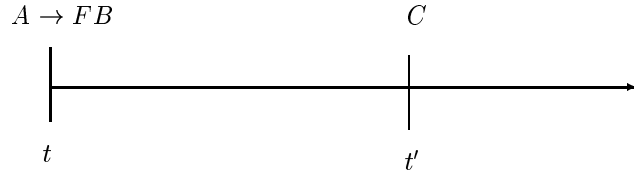


Figure 5.

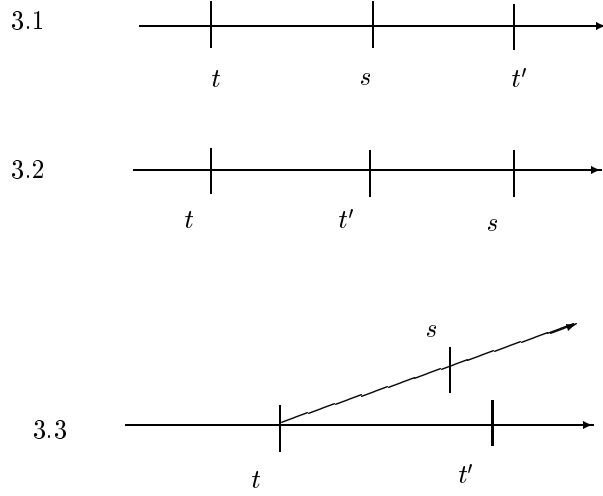


Figure 6.

If time is linear, configuration 3.3, shown in Fig. 6, does not arise and we are essentially checking 3.1, 3.2 of Fig. 6 and the case 4a corresponding to $t' = s$.

If we do not have any special properties of time, success in case 3.2 is required. Since we must succeed in all cases and 3.2 is the case with *least* assumptions, it is enough to check 3.2 alone.

Thus for the case of no special properties of the flow of time, case 4 can be replaced by case 4 general below:

4 **general** $\mathbf{S}(\rho, <, \Delta, FG, t, G_0, t_0) \in \Pi$ and for the future configuration $(\rho_1, <_1)$ defined as $\rho_1 = \rho \cup \{s\}$ and $<_1 = < \cup \{t < s\}$, s a new letter, we have that: $\Pi's = (\Pi - \{\mathbf{S}(\rho, <, \Delta, FG, t, G_0, t_0)\}) \cup \{\mathbf{S}(\rho, <, \Delta, E\Theta_1, t, G_0, t_0)\} \cup \{\mathbf{S}(\rho_1, <_1, \Delta \cup \{(s : B_j) \mid j = 1, 2, \dots\}, D, s, G_0, t_0)\}$ succeeds with $\Theta'_s = \Theta\Theta_1$ and $D = G \vee FG$ and $E = A$.

4*. **Second configuration rule for F :**

For some $\mathbf{S}(\rho, <, \Delta, FG, t, G_0, t_0)$ and some $\Box(A \rightarrow F \wedge_j B_j) \in \Delta$ and some $\Theta_1 \Theta_2$ we have both cases 4a and 4b above true with $E = A \vee FA$ and $D = G \vee FG$.

4* **general** Similar to (4 **general**) for the case of general flow.

5. This is the the mirror image of 3 with ' PG ' replacing ' FG ' and ' $s < t$ ' replacing ' $t < s$ '.

6; 6* This is the mirror image of 4 and 4* with ' PG ' replacing ' FG ', ' $s < t$ ' replacing ' $t < s$ ' and 'past configuration' replacing 'future configuration'.

6 **general** This is the image of 4 **general**.

We now give the computation rules 7-10 for $\bigcirc$ and $\bullet$ for orderings in which a next point and/or previous points exist. If $t \in T$ has a next point we denote this point by $s = \bigcirc t$. If it has a previous point we denote it by $s = \bullet t$. For example, if $(T, <)$ is the integers then $\bigcirc t = t + 1$ and $\bullet t = t - 1$. If $(T, <)$ is a tree then $\bullet t$ always exists, except at the root, but $\bigcirc t$ may or may not exist. For the sake of simplicity we must assume that if we have $\bigcirc$ or $\bullet$ in the language then $\bigcirc t$ or $\bullet t$ always exist. Otherwise we can sneak negation in by putting $(t : \bigcirc A) \in \Delta$ when $\bigcirc t$ does not exist!

7. **Immediate rule for $\bigcirc$:**

$\mathbf{S}(\rho, <, \Delta, \bigcirc G, t, G_0, t_0) \in \Pi$ and $\bigcirc t$ exists and $\bigcirc t \in \rho$ and $\Theta' = \Theta$ and $\Pi' = (\Pi - \{\mathbf{S}(\rho, <, \Delta, \bigcirc G, t, G_0, t_0)\}) \cup \{\mathbf{S}(\rho, <, \Delta, G, \bigcirc t, G_0, t_0)\}$.

8. **Configuration rule for $\bigcirc$:**

$\mathbf{S}(\rho, <, \Delta, \bigcirc G, t, G_0, t_0) \in \Pi$ and for some Θ_1, Θ_2 some $(t : A \rightarrow \bigcirc \wedge_j B_j) \in \Delta$ and $\Pi' = (\Pi - \{\mathbf{S}(\rho, <, \Delta, \bigcirc G, t, G_0, t_0)\}) \cup \{\mathbf{S}(\rho, <, \Delta, A\Theta_1, t, G_0, t_0)\} \cup \{\mathbf{S}(\rho \cup \{\bigcirc t\}, <', \Delta \cup \{(\bigcirc t : B_j)\}, G, \bigcirc t, G_0, t_0)\}$ succeeds with $\Theta' = \Theta\Theta_2$, and $<'$ is the appropriate ordering closure of $< \cup \{(t, \bigcirc t)\}$.

Notice that case 8 is parallel to case 4. We do not need 8a and 8b because of $\bigcirc t \in \rho$; then what would be case 8b becomes 7.

9. The mirror image of 7 with ' $\bullet$ ' replacing ' $\bigcirc$ '.

10. The mirror image of 8 with ' $\bullet$ ' replacing ' $\bigcirc$ '.

11. Negation as failure rule:

$\mathbf{S}(\rho, <, \Delta, \neg G, t, G_0, t_0) \in \Pi$ and Θ grounds every type 2 variable and the computation for success of $\mathbf{S}(\rho, <, \Delta, G, t, \Theta)$ ends up in failure.

12. Disjunction rule:

$\mathbf{S}(\rho, <, \Delta, G_1 \vee G_2, t, G_0, t_0) \in \Pi$ and $\Pi' = (\Pi - \{\mathbf{S}(\rho, <, \Delta, G_1 \vee G_2, t, G_0, t_0)\}) \cup \{\mathbf{S}(\rho, <, \Delta, G_i, t, G_0, t_0) \mid i \in \{1, 2\}\}$ and $\Theta' = \Theta$ and $i \in \{1, 2\}$.

13. Conjunction rule:

$\mathbf{S}(\rho, <, \Delta, G_1 \wedge G_2, t, G_0, t_0) \in \Pi$ and $\Pi' = (\Pi - \{\mathbf{S}(\rho, <, \Delta, G_1 \wedge G_2, t, G_0, t_0)\}) \cup \{\mathbf{S}(\rho, <, \Delta, G_i, t, G_0, t_0) \mid i \in \{1, 2\}\}$.

14. Restart rule:

$\mathbf{S}(\rho, <, \Delta, G, t, G_0, t_0) \in \Pi$ and $\Pi' = (\Pi - \{\mathbf{S}(\rho, <, \Delta, G, t, G_0, t_0)\}) \cup \{\mathbf{S}(\rho, <, \Delta, G_1, t_0, G_0, t_0)\}$ where G_1 is obtained from G_0 by substituting completely new type 2 variables u'_i for the type 2 variables u_i of G_0 , and where Θ' extends Θ by giving $\Theta'(u'_i) = u'_i$ for the new variables u'_i .

15. To start the computation:

Given Δ and $t_0 : G_0$ and a flow $(T, <)$, we start the computation with $\Pi = \{\mathbf{S}(\rho, <, \Delta, G_0, t_0, G_0, t_0)\}$, where $(\rho, <)$ is the configuration associated with Δ , over $(T, <)$ (Definition 75).

Let us check some examples.

EXAMPLE 78.

Data:

1. $t : a \rightarrow Fb$
2. $\Box(b \rightarrow Fc)$
3. $t : a$.

Query: $?t : Fc$

Configuration: $\{t\}$

Using rule 4* we create a future s with $t < s$ and ask the two queries (the notation $A?B$ means that we add A to the data 1, 2, 3 and ask $?B$).

4. $?t : \mathcal{C} \vee Fb$
and
5. $s : c?s : c \vee Fc$

5 succeeds and 4 splits into two queries by rule 4.

6. $?t : a$
and
7. $s' : b ? s' : b$.

EXAMPLE 79.

Data:

1 $t : FA$

2 $t : FB$

Query: $t : F\varphi$ where $\varphi = (A \wedge B) \vee (A \wedge FB) \vee (B \wedge FA)$.

The query will fail in any flow of time in which the future is not linear. The purpose of this example is to examine what happens when time is linear. Using 1 we introduce a point s , with $s : A$, and query from s the following:

$$?s : \varphi \vee F\varphi$$

If we do not use the restart rule, the query will fail. Now that we are at a point s there is no way to go back to t . We therefore cannot reason that we also have a point $s' : B$ and $t < s$ and $t < s'$ and that because of linearity $s = s'$ or $s < s'$ or $s' < s$. However, if we are allowed to restart, we can continue and ask $t : F\varphi$ and now use the clause $t : FB$ to introduce s' . We now reason using linearity in rule 4 that the configurations are:

$$\begin{aligned} & t < s < s' \\ \text{or } & t < s' < s \\ \text{or } & t < s = s' \end{aligned}$$

and φ succeeds at t for each configuration.

The reader should note the reason for the need to use the restart rule. When time is just a partial order, the two assumptions $t : FA$ and $t : FB$ do not interact. Thus when asking $t : FC$, we know that there are two points $s_1 : A$ and $s_2 : B$; see Fig. 7.

C can be true in either one of them. $s_1 : A$ has no influence on $s_2 : B$. When conditions on time (such as linearity) are introduced, s_1 does influence s_2 and hence we must introduce both at the same time. When one does forward deduction one can introduce both s_1 and s_2 going forward. The backward rules do not allow for that. That is why we need the restart rule. When we restart, we keep all that has been done (with, for example, s_1) and have the opportunity to restart with s_2 . The restart rule can be used to solve the linearity problem for classical logic only. Its side-effect is that it turns intuitionistic logic into classical logic; see Gabbay's paper on *N-Prolog* [Gabbay, 1985] and [Gabbay and Olivetti, 2000]. In theorem

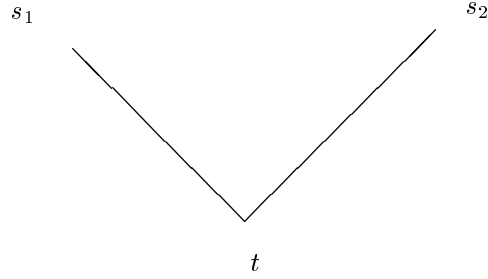


Figure 7.

proving based on intuitionistic logic where disjunctions are allowed, forward reasoning cannot be avoided. See the next example.

It is instructive to translate the above into Prolog and see what happens there.

EXAMPLE 80.

1. $t : FA$ translates into $(\exists s_1 > t)A^*(s_1)$.

2. $t : FB$ translates into $(\exists s_2 > t)A^*(s_2)$.

The query translates into the formula $\psi(t)$:

$$\psi = \exists s > t[A^*(s) \wedge B^*(s)] \vee \exists s_1 > t[A^*(s_1) \wedge \exists s_2 > s_1 B^*(s_2)] \vee \exists s_2 > t[B^*(s_2) \wedge \exists s_1 > s_2 A^*(s_2)]$$

which is equivalent to the disjunction of:

- (a) $[t < s \wedge A^*(s) \wedge B^*(s)]$;
- (b) $t < s_1 \wedge s_1 < s_2 \wedge A^*(s_1) \wedge B^*(s_2)$;
- (c) $t < s_1 \wedge s_2 < s_1 \wedge A^*(s_1) \wedge B^*(s_2)$.

All of (a), (b), (c) fail from the data, unless we add to the data the disjunction

$$\forall xy(x < y \vee x = y \vee y < x).$$

Since this is not a Horn clause, we are unable to express it in the database.

The logic programmer might add this as an integrity constraint. This is wrong as well. As an integrity constraint it would require the database to indicate which of the three possibilities it adopts, namely:

- $x < y$ is in the data;
- or $x = y$ is in the data;
- or $y < x$ is in the data.

This is stronger than allowing the disjunction in the data.

The handling of the integrity constraints corresponds to our metahandling of what configurations $(\rho, <)$ are allowed depending on the ordering of time. By labelling data items we are allowing for the metalevel considerations to be done separately on the labels.

This means that we can handle properties of time which are not necessarily expressible by an object language formula of the logic. In some cases (finiteness of time) this is because they are not first order; in other cases (irreflexivity) it is because there is no corresponding formula (axiom) and in still other cases because of syntactical restrictions (linearity).

We can now make clear our classical versus intuitionistic distinction. If the underlying logic is classical then we are checking whether $\Delta \vdash \psi$ in classical logic. If our underlying logic is intuitionistic, then we are checking whether $\Delta \vdash \psi$ in intuitionistic logic where Δ and ψ are defined below.

Δ is the translation of the data together with the axioms for linear ordering, i.e. the conjunction of:

1. $\exists s_1 > tA^*(s_1)$;
2. $\exists s_2 > tB^*(s_2)$;
3. $\forall xy(x < y \vee x = y \vee y < x)$;
4. $\forall x\exists y(x < y)$;
5. $\forall x\exists y(y < x)$;
6. $\forall xyz(x < y \wedge y < z \rightarrow x < z)$;
7. $\forall x\neg(x < x)$.

ψ is the translation of the query as given above.

The computation of Example 79, using restart, answers the question $\Delta \vdash ?\psi$ in classical logic. To answer the question $\Delta \vdash ?\psi$ in intuitionistic logic we cannot use restart, but must use forward rules as well.

EXAMPLE 81. See Example 79 for the case that the underlying logic is intuitionistic. *Data* and *Query* as in Example 79.

Going forward, we get:

- 3 $s : A$ from 1;
- 4 $s' : B$ from 2.

By linearity,

$$t < s < s'$$

or $t < s' < s$

or $t < s = s'$.

ψ will succeed for each case.

Our language does not allow us to ask queries of the form $\Box G(x)$, where x are all universal variables (i.e. $\forall x \Box G(x)$). However, such queries can be computed from a database Δ . The *only* way to get *always* information out of Δ for a general flow of time is via the always clauses in the database. Always clauses are true everywhere, so if we want to know what else is true everywhere, we ask it from the always clauses. Thus to ask

$? \Box G(x), x$ a universal variable

we first Skolemize and then ask

$\{X, \Box X \mid \Box X \in \Delta\} ? G(c)$

where c is a Skolem constant.

We can add a new rule to Definition 76:

16. Always rule:

$\mathbf{S}(\rho, <, \Delta, \Box G, t, G_0, t_0) \in \Pi$ and

$\Pi' = (\Pi - \{\mathbf{S}(\rho, <, \Delta, \Box G, t, G_0, t_0)\}) \cup \{\mathbf{S}(\{s\}, \emptyset, \Delta', G', s, g', s)\}$

where s is a completely new point and G' is obtained from G by substituting new Skolem constants for all the universal variables of G and

$$\Delta' = \{B, \Box B \mid \Box B \in \Delta\}.$$

We can use 16 to add another clause to the computation of Definition 76, namely:

17. $\mathbf{S}(\rho, <, \Delta, F(A \wedge B), t, G_0, t_0) \in \Pi$

and $\Pi' = (\Pi - \{\mathbf{S}(\rho, <, \Delta, F(A \wedge B), t, G_0, t_0)\}) \cup$

$\{\mathbf{S}(\rho, <, \Delta, FA, t, G_0, t_0), \mathbf{S}(\rho, <, \Delta, \Box B, t, G_0, t_0)\}.$

EXAMPLE 82.

Data	Query	Configuration
$\Box a$	$t : F(a \wedge b)$	$\{t\}$
$t : Fb$		

First computation

Create $s, t < s$ and get

Data	Query	Configuration
$\Box a$	$s : a \wedge b$	$t < s$
$t : Fb$		
$s : b$		

$s : b$ succeeds from the data. $s : a$ succeeds by rule 2, Definition 76.

Second computation

Use rule 17. Since $? \Box a$ succeeds ask for Fb and proceed as in the first computation.

6.3 Different flows of time

We now check the effect of different flows of time on our logical deduction (computation). We consider a typical example.

EXAMPLE 83.

Data	Query	Configuration
$t : FFA$	$?t : FA$	$\{t\}$

The possible world flow is a general binary relation.

We create by rule 4b of Definition 76 a future configuration $t < s$ and add to the database $s : FA$. We get

Data	Query	Configuration
$t : FFA$	$?t : FA$	$t < s$
$s : FA$		

Again we apply rule 4a of Definition 76 and get the new configuration with $s < s'$ and the new item of data $s' : A$. We get

Data	Query	Configuration
$t : FFA$	$?t : FA$	$t < s$
$s : FA$		$s < s'$
$s' : A$		

Whether or not we can proceed from here depends on the flow of time. If $<$ is transitive, then $t < s'$ holds and we can get $t : FA$ in the data by rule 3.

Actually by rule 4* we could have proceeded along the following sequence of deduction. Rule 4* is especially geared for transitivity.

Data	Query	Configuration
$t : FFA$	$t : FA$	t

Using rule 4* we get

Data	Query	Configuration
$t : FFA$	$s : FA \vee FFA$	$t < s$
$s : FA$		

The first disjunct of the query succeeds.

If $<$ is not transitive, rule 3 does not apply, since $t < s'$ does not hold.

Suppose our query were $?t : FFFA$.

If $<$ is reflexive then we can succeed with $?t : FFFA$ because $t < t$.

If $<$ is dense (i.e. $\forall xy(x < y \rightarrow \exists z(x < z \wedge z < y))$) we should also succeed because we can create a point z with $t < z < s$.

$z : FFA$ will succeed and hence $t : FFA$ will also succeed.

Here we encounter a new rule (density rule), whereby points can always be ‘landed’ between existing points in a configuration.

We now address the flow of time of the type natural numbers, $\{1, 2, 3, 4, \dots\}$. This has the special property that it is generated by a function symbol s :

$$\{1, s(1), ss(1), \dots\}.$$

EXAMPLE 84.

Data	Query	Configuration
$\Box(q \rightarrow \bigcirc q)$ $1 : \bigcirc q$ $1 : Fp$	$1 : F(p \wedge q)$	$\{1\}$

If time is the natural numbers, the query should succeed from the data. If time is not the natural numbers but, for example, $\{1, 2, 3, \dots, w, w+1, w+2, \dots\}$ then the query should fail.

How do we represent the fact that time is the natural numbers in our computation rule? What is needed is the ability to do some induction. We can use rule 4b and introduce a point t with $1 < t$ into the configuration and even say that $t = n$, for some n . We thus get

Data	Query	Configuration
$\Box(q \rightarrow \bigcirc q)$ $1 : \bigcirc q$ $1 : Fp$ $n : p$	$1 : F(p \wedge q)$	$1 < n$

Somehow we want to derive $n : q$ from the first two assumptions. The key reason for the success of $F(p \wedge q)$ is the success of $\Box q$ from the first two assumptions. We need an induction axiom on the flow of time.

To get a clue as to what to do, let us see what Prolog would do with the translations of the data and goal.

Translated data

$$\begin{aligned} &\forall t[1 \leq t \wedge Q^*(t) \rightarrow Q^*(t+1)] \\ &Q^*(1) \\ &\exists t P^*(t). \end{aligned}$$

Translated query

$$\exists t(P^*(t) \wedge Q^*(t)).$$

After we Skolemize, the database becomes:

1. $1 \leq t \wedge Q^*(t) \rightarrow Q^*(t+1)$
2. $Q^*(1)$
3. $P^*(c)$

and the query is

$$P^*(s) \wedge Q^*(s).$$

We proceed by letting $s = c$. We ask $Q^*(c)$ and have to ask after a slightly generalized form of unification $?1 \leq c \wedge Q^*(c-1)$.

Obviously this will lead nowhere without an induction axiom. The induction axiom should be that for *any* predicate $PRED$

$$PRED(1) \wedge \forall x[1 \leq x \wedge PRED(x) \rightarrow PRED(x+1)] \rightarrow \forall x PRED(x).$$

Written in Horn clause form this becomes

$$\begin{aligned} \exists x \forall y [& PRED(1) \wedge [1 \leq x \wedge PRED(x) \\ & \rightarrow PRED(x+1)] \rightarrow PRED(y)]. \end{aligned}$$

Skolemizing gives us

$$4. \ PRED(1) \wedge (1 \leq d \wedge PRED(d) \rightarrow PRED(d+1)) \rightarrow PRED(y)$$

where d is a Skolem constant.

Let us now ask the query $P^*(s) \wedge Q^*(s)$ from the database with 1–4. We unify with clause 3 and ask $Q^*(c)$. We unify with clause 4 and ask $Q^*(1)$ which succeeds and ask for the implication

$$?1 \leq d \wedge Q^*(d) \rightarrow Q^*(d+1).$$

This should succeed since it is a special case of clause 1 for $t = d$.

The above shows that we need to add an induction axiom of the form

$$\bigcirc x \wedge \Box(x \rightarrow \bigcirc x) \rightarrow \Box x.$$

Imagine that we are at time t , and assume $t' < t$. If A is true at t' and $\Box(A \rightarrow \bigcirc A)$ is true, then A is true at t .

We thus need the following rule:

18. Induction rule:

$t : F(A \wedge B)$ succeeds from Δ at a certain configuration if the following conditions all hold.

1. $t : FB$ succeed.
2. For some $s < t, s : A$ succeeds.

3. $m : \bigcirc A$ succeeds from the database Δ' , where $\Delta' = \{X, \Box X \mid \Box X \in \Delta\} \cup \{A\}$ and m is a completely new time point and the new configuration is $\{m\}$.

The above shows how to compute when time is the natural number. This is not the best way of doing it. In fact, the characteristic feature involved here is that the ordering of the flow of time is a Herbrand universe generated by a finite set of function symbols. FA is read as ‘ A is true at a point generated by the function symbols’. This property requires a special study. See Chapter 11 of [Gabbay *et al.*, 1994].

6.4 A theorem prover for modal and temporal logics

This section will briefly indicate how our temporal Horn clause computation can be extended to an automated deduction system for full modal and temporal logic. We present computation rules for propositional temporal logic with $F, P, \bigcirc, \bullet, \wedge \rightarrow$ and $\perp$. We intend to approach predicate logic in Volume 3 as it is relatively complex. The presentation will be intuitive.

DEFINITION 85. We define the notions of a *full clause*, a *body* and a *head*.

- (a) A *full clause* is an atom q or $\perp$ or $B \rightarrow H$, or H where B is a body and H is a head.
- (b) A *body* is a conjunction of full clauses.
- (c) A *head* is an atom q or $\perp$ or FH or PH or $\bigcirc H$ or $\bullet H$, where H is a body.

Notice that negation by failure is not allowed. We used the connectives $\wedge, \rightarrow, \perp$. The other connectives, $\vee$ and $\sim$, are definable in the usual way: $\sim A = A \rightarrow \perp$ and $A \vee B = (A \rightarrow \perp) \rightarrow B$. The reader can show that every formula of the language with the connectives $\{\sim, \wedge, \vee, F, G, P, H\}$ is equivalent to a conjunction of full clauses. We use the following equivalences:

$$A \rightarrow (B \wedge C) = (A \rightarrow B) \wedge (A \rightarrow C);$$

$$A \rightarrow (B \rightarrow C) = A \wedge B \rightarrow C;$$

$$GA = F(A \rightarrow \perp) \rightarrow \perp;$$

$$HA = P(A \rightarrow \perp) \rightarrow \perp.$$

DEFINITION 86. A database is a set of labelled full clauses of the form $(\Delta, \rho, <)$, where $\rho = \{t \mid t : A \in \Delta, \text{ for some } A\}$. A query is a labelled full clause.

DEFINITION 87. The following is a definition of the predicate $\mathbf{S}(\rho, <, \Delta, G, t, G_0, t_0)$, which reads: the labelled goal $t : G$ succeeds from $(\Delta, \rho, <)$ with parameter (initial goal) $t_0 : G_0$.

- 1(a) $\mathbf{S}(\rho, <, \Delta, q, t, G_0, t_0)$ for q atomic or $\perp$ if for some $t : A \rightarrow q$, $\mathbf{S}(\rho, <, \Delta, A, t, G_0, t_0)$.
- (b) If $t : q \in \Delta$ or $s : \perp \in \Delta$ then $\mathbf{S}(\rho, <, \Delta, q, t, G_0, t_0)$.
- (c) $\mathbf{S}(\rho, <, \Delta, \perp, t, G_0, t_0)$ if $\mathbf{S}(\rho, <, \Delta, \perp, s, G_0, t_0)$.
This rule says that if we can get a contradiction from any label, it is considered a contradiction of the whole system.
2. $\mathbf{S}(\rho, <, \Delta, G, t, G_0, t_0)$ if for some $s : A \rightarrow \perp$, $\mathbf{S}(\rho, <, \Delta, A, s, G_0, t_0)$.
3. $\mathbf{S}(\rho, <, \Delta, t, FG, G_0, t_0)$ if for some $s \in \rho, t < s$ and $\mathbf{S}(\rho, <, \Delta, G, s, G_0, t_0)$.
4. $\mathbf{S}(\rho, <, \Delta, FG, t, G_0, t_0)$ if for some $t : A \rightarrow FB \in \Delta$ we have that both (a) and (b) below hold true:
 - (a) For all $s \in \rho$ such that $t < s$ we have $\mathbf{S}(\rho, <, \Delta^*, s, D, G_0, t_0)$ and $\mathbf{S}(\rho, <, \Delta, E, t, G_0, t_0)$ hold, where $\Delta^* = \Delta \cup \{s : B\}$ and $D \in \{G, FG\}$ and $E \in \{A, FA\}$.
Note: The choice of D and E is made here for the case of transitive time. In modal logic, where $<$ is not necessarily transitive, we take $D = G, E = A$. Other conditions on $<$ correspond to different choices of D and E .
 - (b) For all future configurations of $(\rho, <, t)$ with a new letter s , denoted by $(\rho_s, <_s)$, we have $\mathbf{S}(\rho_s, <_s, \Delta^*, s, D, G_0, t_0)$ and $\mathbf{S}(\rho_s, <_s, \Delta, E, t, G_0, t_0)$ hold, where Δ^*, E, D are as in (a).
5. This is the mirror image of 3.
6. This is the mirror image of 4.
- 7(a) $\mathbf{S}(\rho, <, \Delta, A_1 \wedge A_2, t, G_0, t_0)$ if both $\mathbf{S}(\rho, <, \Delta, A_i, t, G_0, t_0)$ hold for $i = 1, 2$.
- (b) $\mathbf{S}(\rho, <, \Delta, A \rightarrow B, t, G_0, t_0)$ if $\mathbf{S}(\rho, <, \Delta \cup \{t : A\}, B, t, G_0, t_0)$.
8. **Restart rule:**
 $\mathbf{S}(\rho, <, \Delta, G, t, G_0, t_0)$ if $\mathbf{S}(\rho, <, \Delta, G_0, t_0, G_0, t_0)$.

If the language contains $\bigcirc$ and $\bullet$ then the following are the relevant rules.

9. $\mathbf{S}(\rho, <, \Delta, \bigcirc G, t, G_0, t_0)$ if $\bigcirc t$ exists and $\bigcirc t \in \rho$ and $\mathbf{S}(\rho, <, \Delta, G, \bigcirc t, G_0, t_0)$.
10. $\mathbf{S}(\rho, <, \Delta, \bigcirc G, t, G_0, t_0)$ if for some $t : A \rightarrow \bigcirc B \in \Delta$ both $\mathbf{S}(\rho, <, \Delta, A, t, G_0, t_0)$ and $\mathbf{S}(\rho \cup \{\bigcirc t\}, <', \Delta \cup \{\bigcirc t : B\}, G, \bigcirc t, G_0, t_0)$ hold where $<'$ is the appropriate ordering closure of $< \cup \{t < \bigcirc t\}$.

11. This is the mirror image of 9 for $\bullet$.

12. This is the mirror image of 10 for $\bullet$.

EXAMPLE 88. (Here $\square$ can be either G or H .)

	Data	Query	Configuration
1.	$t : \square a$	$?t : \square b$	$\{t\}$
2.	$t : \square(a \rightarrow b)$		t is a constant

Translation:

	Data	Query	Configuration
1.	$t : F(a \rightarrow \perp) \rightarrow \perp$	$t : F(b \rightarrow \perp) \rightarrow \perp$	$\{t\}$
2.	$t : F((a \rightarrow b) \rightarrow \perp) \rightarrow \perp$		

Computation

The problem becomes

	Additional data	Current query	Configuration
3.	$t : F(b \rightarrow \perp)$	$?t : \perp$	$\{t\}$
	from 2	$?t_0 : F((a \rightarrow b) \rightarrow \perp)$	

From 3 using ** create a new point s :

	Additional data	Current query	Configuration
4.	$s : b \rightarrow \perp$	$?s : (a \rightarrow b) \rightarrow \perp$	$t < s$
	Add $s : a \rightarrow b$ to the database and ask		
5.	$s : (a \rightarrow b)$	$?s : \perp$	

From 4 and 5 we ask:

$?s : a$.

From computation rule 2 and clause 1 of the data we ask

$?t : F(a \rightarrow \perp)$.

From computation rule 2 we ask

$?s : a \rightarrow \perp$

We add $s : a$ to the data and ask

	Additional data	Current query	Configuration
6.	$s : a$	$?s : \perp$	$t < s$

The query succeeds.

6.5 Modal and temporal Herbrand universes

This section deals with the soundness of our computation rules. In conjunction with soundness it is useful to clarify the notion of modal and temporal Herbrand models. For simplicity we deal with temporal logic with P, F only and transitive irreflexive time or with modal logic with one modality $\Diamond$ and a general binary accessibility relation $<$. We get our clues from some examples:

EXAMPLE 89. Consider the database

1. $t : a \rightarrow \Diamond b$
2. $\Box(b \rightarrow c)$
3. $t : a.$

The constellation is $\{t\}$.

If we translate the clauses into predicate logic we get:

1. $a^*(t) \rightarrow \exists s > tb^*(s)$
2. $\forall x[b^*(x) \rightarrow c^*(x)]$
3. $a^*(t).$

Translated into Horn clauses we get after Skolemising:

- 1.1 $a^*(t) \rightarrow b^*(s)$
- 1.2 $a^*(t) \rightarrow t < s$
- 2 $b^*(x) \rightarrow c^*(x)$
- 3 $a^*(t).$

t, s are Skolem constants.

From this program, the queries

$$a^*(t), \neg b^*(t), \neg c^*(t), \neg a(s), b(s), c^*(s)$$

all succeed. $\neg$ is negation by failure.

It is easy to recognize that $\neg a^*(s)$ succeeds because there is no head which unifies with $a^*(s)$. The meaning of the query $\neg a^*(s)$ in terms of modalities is the query $\Diamond \neg a$.

The question is: how do we recognize syntactically what fails in the modal language? The heads of clauses can be whole databases and there is no immediate way of syntactically recognizing which atoms are not heads of clauses.

EXAMPLE 90. We consider a more complex example:

1. $t : a \rightarrow \Diamond b$
2. $\Box(b \rightarrow c)$
3. $t : a$
4. $t : a \rightarrow \Diamond d$.

We have added clause 4 to the database in the previous example. The translation of the first three clauses will proceed as before. We will get

- 1.1 $a^*(t) \rightarrow c^*(s)$
- 1.2 $a^*(t) \rightarrow t < s$
- 2 $b^*(x) \rightarrow c^*(x)$
- 3 $a^*(t)$.

We are now ready to translate clause 4. This should be translated like clause 1 into

- 4.1 $a^*(t) \rightarrow d^*(r)$
- 4.2 $a^*(t) \rightarrow t < r$.

The above translation is correct if the set of possible worlds is just an ordering. Suppose we know further that in our modal logic the set of possible worlds is linearly ordered. Since $t < s \wedge t < r \rightarrow s = r \vee s < r \vee r < s$, this fact must be reflected in the Horn clause database. The only way to do it is to add it as an integrity constraint.

Thus our temporal program translates into a Horn clause program with integrity constraints.

This will be true in the general case. Whether we need integrity constraints or not will depend on the flow of time.

Let us begin by translating from the modal and temporal language into Horn clauses. The labelled wff $t : A$ will be translated into a set of formulae of predicate logic denoted by $Horn(t, A)$. $Horn(t, A)$ is supposed to be logically equivalent to A . The basic translation of a labelled atomic predicate formula $t : A(x_1 \dots x_n)$ is $A^*(t, x_1 \dots x_n)$. A^* is a formula of a two-sorted predicate logic where the first sort ranges over labels and the second sort over domain elements (of the world t).

DEFINITION 91. Consider a temporal predicate language with connectives P and F , and $\neg$ for negation by failure.

Consider the notion of labelled temporal clauses, as defined in Definition 73.

Let $Horn(t, A)$ be a translation function associating with each labelled clause or goal a set of Horn clauses in the two-sorted language described above. The letters t, s which appear in the translation are Skolem constants. They are assumed to be *all different*.

We assume that we are dealing with a general transitive flow of time. This is to simplify the translation. If time has extra conditions, i.e. linearity, additional integrity constraints may need to be added. If time is characterized by non-first-order conditions (e.g. finiteness) then an adequate translation into Horn clause logic may not be possible.

The following are the translation clauses:

1. $Horn(t, A(x_1 \dots x_n)) = A^*(t, x_1 \dots x_n)$, for A atomic;
2. $Horn(t, FA) = \{t < s\} \cup Horn(s, A)$
 $Horn(t, PA) = \{s < t\} \cup Horn(s, A)$;
3. $Horn(t, A \wedge B) = Horn(t, A) \cup Horn(t, B)$;
4. $Horn(t, \neg A) = \neg \bigwedge Horn(t, A)$;
5. $Horn(t, A \rightarrow F \wedge B_j) = \{\bigwedge Horn(t, A) \rightarrow t < s\} \cup \bigcup_{B_j} \{\bigwedge Horn(s, A) \wedge C \rightarrow D \mid (C \rightarrow D) \in Horn(s, B_j)\}$;
6. $Horn(t, A \rightarrow P \wedge B_j) = \{\bigwedge Horn(t, A) \rightarrow s < t\} \cup \bigcup_{B_j} \{\bigwedge Horn(s, A) \wedge C \rightarrow D \mid (C \rightarrow D) \in Horn(s, B_j)\}$;
7. $Horn(t, \Box A) = Horn(x, A)$ where x is a universal variable.

EXAMPLE 92. To explain the translation of $t : A \rightarrow F(B_1 \wedge (B_2 \rightarrow B_3))$, let us write it in predicate logic. $A \rightarrow F(B_1 \wedge (B_2 \rightarrow B_3))$ is true at t if A true at t implies $F(B_1 \wedge (B_2 \rightarrow B_3))$ is true at t . $F(B_1 \wedge (B_2 \rightarrow B_3))$ is true at t if for some s , $t < s$ and $B_1 \wedge (B_2 \rightarrow B_3)$ are true at s .

Thus we have the translation

$$A^*(t) \rightarrow \exists s(t < s \wedge B_1^*(s) \wedge (B_2^*(s) \rightarrow B_3^*(s))).$$

Skolemizing on s and writing it in Horn clauses we get the conjunction

$$\begin{aligned} A^*(t) &\rightarrow t < s \\ A^*(t) &\rightarrow B_1^*(s) \\ A^*(t) \wedge B_2^*(s) &\rightarrow B_3^*(s). \end{aligned}$$

Let us see what the translation $Horn$ does:

$$\begin{aligned} Horn(t, A \rightarrow F(B_1 \wedge (B_2 \rightarrow B_3))) &= \{\bigwedge Horn(t, A) \rightarrow t < s\} \cup \\ &\{\bigwedge Horn(t, A) \rightarrow Horn(s, B_2)\} \cup \{\bigwedge Horn(t, A) \wedge \bigwedge Horn(s, B_2) \rightarrow \\ &\bigwedge Horn(s, B_3)\} = \{A^*(t) \rightarrow t < s, A^*(t) \rightarrow B_2^*(s), A^*(t) \wedge B_2^*(s) \rightarrow B_3^*(s)\}. \end{aligned}$$

We prove soundness of the computation of Definition 76, relative to the Horn clause computation for the Horn database in classical logic. In other words, if the translation $Horn(t, A)$ is accepted as sound, as is intuitively clear, then the computation of $\mathbf{S}(\rho, <, \Delta, G, t, G_0, t_0, \Theta)$ can be translated isomorphically into a classical Horn clause computation of the form $Horn(t, \Delta)?Horn(t, G)$, and the soundness of the classical Horn clause computation would imply the soundness of our computation.

This method of translation will also relate our temporal computation to that of an ordinary Horn clause computation.

The basic unit of our temporal computation is $\mathbf{S}(\rho, <, \Delta, G, t, G_0, t_0, \Theta)$. The current labelled goal is $t : G$ and $t_0 : G_0$ is the original goal. The database is $(\rho, <, \Delta)$ and Θ is the current substitution. $t_0 : G_0$ is used in the restart rule. For a temporal flow of time which is ordinary transitive $<$, we do not need the restart rule. Thus we have to translate $(\rho, <, \Delta)$ to classical logic and translates $t : G$ and Θ to classical logic and see what each computation step of $\mathbf{S}$ of the source translates into the classical logic target.

DEFINITION 93. Let $(\rho, <)$ be a constellation and let Δ be a labelled database such that

$$\rho = \{t \mid \text{for some } A, t : A \in \Delta\}.$$

Let $Horn((\rho, <), \Delta) = \{t < s \mid t, s \in \rho \text{ and } t < s\} \cup \bigcup_{t:A \in \Delta} Horn(t, A)$.

THEOREM 94 (Soundness). $\mathbf{S}(\rho, <, \Delta, G, t, \Theta)$ *succeeds in temporal logic if and only if in the sorted classical logic $Horn((\rho, <), \Delta)?Horn(t, G)$ succeeds with Θ .*

Proof. The proof is by induction on the complexity of the computation tree of $\mathbf{S}(\rho, <, \Delta, G, t, \Theta)$.

We follow the inductive steps of Definition 75. The translation of (Π, Θ) is a conjunction of Horn clause queries, all required to succeed under the same substitution Θ .

Case I The empty goal succeeds in both cases.

Case II (Π, Θ) fails if for some $\mathbf{S}(\rho, <, \Delta, G, t)$, we have G is atomic and for all $\Box(A \rightarrow H) \in \Delta$ and all $t : A \rightarrow H \in \Delta$, $G\Theta$ and $H\Theta$ do not unify. The reason they do not unify is because of what Θ substitutes to the variables u_i .

The corresponding Horn clause predicate programs are

$$\bigwedge Horn(x, A) \rightarrow H^*(x)$$

and

$$\bigwedge Horn(t, A) \rightarrow H^*(t)$$

and the goal is $?G^*(t)$.

Clearly, since x is a general universal variable, the success of the two-sorted unification depends on the other variables and Θ . Thus unification does *not* succeed in the classical predicate case iff it does not succeed in the temporal case.

Rules 1 and 2 deal with the atomic case: the query is $G^*(t)$ and in the database among the data are

$$\bigwedge \text{Horn}(t, A) \rightarrow H^*(t) \text{ and } \bigwedge \text{Horn}(x, A) \rightarrow H^*(x)$$

for the cases of $t : A \rightarrow H$ and $\Box(A \rightarrow H)$ respectively.

For the Horn clause program to succeed $G^*(t)$ must unify with $H^*(t)$. This will hold if and only if the substitution for the domain variables allows unification, which is exactly the condition of Definition 75.

Rules 3, 4(general) and 4*(general) deal with the case of a goal of the form $?t : FG$. The translation of the goal is $t < u \wedge \bigwedge \text{Horn}(u, G)$ where u is an existential variable.

Rule 3 gives success when for some $s, t < s \in \Delta$ and $?s : G$ succeeds. In this case let $u = s$; then $t < u$ succeeds and $\bigwedge \text{Horn}(s, G)$ succeeds by the induction hypothesis.

We now turn to the general rules 4(general) and 4*(general). These rules yield success when for some clause of the form

$$t : A \rightarrow F \wedge B_j$$

or

$$\Box(A \rightarrow F \wedge B_j).$$

$\Delta ?t : A$ succeeds and $\Delta \cup \{(s : B_j)\} ?s : G \vee FG$ both succeed. s is a new point.

The translation $\bigwedge \text{Horn}(t, A)$ succeeds by the induction hypothesis.

The translation of

$$t : A \rightarrow F \wedge B_j$$

or

$$\Box(A \rightarrow F \wedge B_j)$$

contains the following database:

1. $\bigwedge \text{Horn}(t, A) \rightarrow t < s$.
2. For every B_j and every $C \rightarrow D$ in $\text{Horn}(s, B_j)$ the clause $\bigwedge \text{Horn}(s, A) \wedge C \rightarrow D$.

Since $\bigwedge \text{Horn}(t, A)$ succeeds we can assume we have in our database:

$$1^* \ t < s;$$

2* $C \rightarrow D$, for $C \rightarrow D \in \text{Horn}(s, B_j)$ for some j .

These were obtained by substituting *truth* in 1 and 2 for $\bigwedge \text{Horn}(t, A)$.

The goal is to show $t < u \wedge \bigwedge \text{Horn}(u, G)$.

Again for $u = s$, $t < u$ succeeds from (1*) and by the induction hypothesis, since $\Delta \cup \{s : B_j\} ? s : G \vee FG$ is successful, we get

$$\bigcup_j \text{Horn}(s, B_j) ? \bigwedge \text{Horn}(s, G) \vee (s < u' \wedge \bigwedge \text{Horn}(u', G))$$

should succeed, with u' an existential variable.

However, 2* is exactly $\bigcup_j \text{Horn}(s, \bigwedge B_j)$. Therefore we have shown that rules 4(general) and 4*(general) are sound.

Rules 6(general) and 6*(general) are sound because they are the mirror images of 4(general) and 4*(general).

The next relevant rules for our soundness cases are 11–13. These follow immediately since the rules for $\wedge, \vee, \neg$ are the same in both computations.

Rule 14, the restart rule, is definitely sound. If we try to show in general that $\Delta \vdash A$ then since in classical logic $\sim A \rightarrow A$ is the same as A ($\sim$ is classical negation) it is equivalent to show $\Delta, \sim A \vdash A$.

If $\sim A$ is now in the data, we can *at any time* try to show A instead of the current goal G . This will give us A (shown) and $\sim A$ (in Data) which is a contradiction, and this yields *any goal* including the current goal G .

We have thus completed the soundness proof. $\blacksquare$

6.6 Tractability and persistence

We defined a temporal database Δ essentially as a finite piece of information telling us which temporal formulae are true at what times. In the most general case, for a general flow of time $(T, <)$, all a database can do is to provide a set of the form $\{t_i : A_i\}$, meaning that A_i is true at time t_i and a configuration $(\{t_i\}, <)$, giving the temporal relationships among $\{t_i\}$. A query would be of the form $?t : Q$, where t is one of the t_i . The computation of the query from the data is in the general case exponential, as we found in Section 6.2, from the case analysis of clause 4 of Definition 76 and from Example 72. We must therefore analyse the reasons for the complexity and see whether there are simplifying natural assumptions, which will make the computational problem more tractable.

There are three main components which contribute to complexity:

1. The complexity of the temporal formulae allowed in the data and in the query. We allow $t : A$ into the database, with A having temporal operators. So, for example, $t : FA$ is allowed and also $t : \bigcirc A$. $t : FA$ makes life more difficult because it has a hidden Skolem function in it. It really means $\exists s[t < s \text{ and } (s : A)]$. This gives rise to case

analysis, as we do not know in general where s is. See Example 80 and Examples 89 and 90. In this respect $t : \bigcirc A$ is a relatively simple item. It says $(t+1) : A$. In fact any temporal operator which specifies the time is relatively less complex. In practice, we do need to allow data of the form $t : FA$. Sometimes we know an event will take place in the future but we do not know when. The mere fact that A is going to be true can affect our present actions. A concrete example where such a case may arise is when someone accepts a new appointment beginning next year, but has not yet resigned from their old position. We know they are going to resign but we do not know when;

2. The flow of time itself gives rise to complexity. The flow of time may be non-Horn clause (e.g. linear time which is defined by a disjunctive axiom

$$\forall xy[x < y \vee y < x \vee x = y].$$

This complicates the case analysis of 1 above.

3. Complexity arises from the behaviour. If atomic predicates get truth values at random moments of time, the database can be complex to describe. A very natural simplifying assumption in the case of temporal logic is *persistence*. If atomic statements and their negations remain true for a while then they give rise to less complexity. Such examples are abundant. For example, people usually stay at their residences and jobs for a while. So for example, any payroll or local tax system can benefit from persistence as a simplifying assumption. Thus in databases where there is a great deal of persistence, we can use this fact to simplify our representation and querying. In fact, we shall see that a completely different approach to temporal representation can be adopted when one can make use of persistence.

Another simplifying assumption is *recurrence*. Saturdays, for example, recur every week, so are paydays. This simplifies the representation and querying. Again, a payroll system would benefit from that.

We said at the beginning that a database Δ is a finitely generated piece of temporal information stating what is true and when. If we do not have any simplifying assumptions, we have to represent Δ in the form $\Delta = \{t_i : A_i\}$ and end up needing the computation rules of Section 6.2 to answer queries.

Suppose now that we adopt all three simplifying assumptions for our database. We assume that the A_i are only atoms and their negations, we further assume that each A_i is either persistent or recurrent, and let us assume, to be realistic, that the flow of time is linear. Linearity does not make the computation more complicated in this particular case, because we are not allowing data of the form $t : FA$, and so complicated case analysis

does not arise. In fact, together with persistence and recurrence, linearity becomes an additional simplifying assumption!

Our aim is to check what form our temporal logic programming machine should take in view of our chosen simplifying assumptions.

First note that the most natural units of data are no longer of the form:

$$t : A$$

reading A is true at t , but either of the form

$$[t, s] : A, [t < s]$$

reading A is true in the closed interval $[t, s]$, or the form

$$t \parallel d : A$$

reading A is true at t and recurrently at $t + d, t + 2d, \dots$, that is, every d moments of time.

A is assumed to be a literal (atom or a negation of an atom) and $[t, s]$ is supposed to be a maximal interval where A is true. In $t \parallel d$, d is supposed to be the minimal cycle for A to recur. The reasons for adopting the notation $[t, s] : A$ and $t \parallel d : A$ are not mathematical but simply intuitive and practical. This is the way we think about temporal atomic data when persistence or recurrence is present. In the literature there has been a great debate on whether to evaluate temporal statements at points or intervals. Some researchers were so committed to intervals that they tended, unfortunately, to disregard any system which uses points. Our position here is clear and intuitive. First perform all the computations using intervals. Evaluation at points is possible and trivial. To evaluate $t : A$, i.e. to ask $?t : A$ as a query from a database, compute the (maximal) intervals at which A is true and see whether t is there. To evaluate $[t, s] : A$ do the same, and check whether $[t, s]$ is a subset.

The query language is left in its full generality. i.e. we can ask queries of the form $t : A$ where A is unrestricted (e.g. $A = FB$ etc.). It makes sense also to allow queries of the form $[t, s] : A$, although exactly how we are going to find the answer remains to be seen. The reader should be aware that the data representation language and the query language are no longer the same. This is an important factor. There has been a lot of confusion, especially among the AI community, in connection with these matters. We shall see later that as far as computational tractability is concerned, the restriction to persistent data allows one to strengthen the query language to full predicate quantification over time points.

At this stage we might consider allowing recurrence within an interval, i.e. we allow something like

$$'A \text{ is true every } d \text{ days in the interval } [t, s].'$$

We can denote this by

$$[t||d, s] : A$$

meaning A is true at $t, t + d, t + 2d$, as long as $t + nd \leq s, n = 1, 2, 3, \dots$

We may as well equally have recurrent intervals. An example of that would be taking a two-week holiday every year. This we denote by

$$[t, s]||d : A, \quad t < s, \quad (s - t) < d,$$

reading A is true at the intervals $[t, s], [t + d, s + d], [t + 2d, s + 2d]$, etc.

The reader should note that adopting this notation takes us outside the realm of first-order logic. Consider the integer flow of time. We can easily say that q is true at all even numbers by writing $[0, 0]||1$ as a truth set for q and $[1, 1]||1$ as a truth set for $\sim q$ (i.e. q is true at 0 and recurs every 1 unit and $\sim q$ is true at 1 and recurs every 1 unit).

The exact expressive power of this language is yet to be examined. It is connected with the language USF which we meet later.

The above seem to be the most natural options to consider. We can already see that it no longer makes sense to check how the computation rules of Definition 76 simplify for our case. Our case is so specialized that we may as well devise computation rules especially for it. This should not surprise us. It happens in mathematics all the time. The theory of Abelian groups, for example, is completely different from the theory of semigroups, although Abelian groups are a special case of semigroups. The case of Abelian groups is so special that it does not relate to the general case any more.

Let us go back to the question of how to answer a query from our newly defined simplified databases. We start with an even more simple case, assuming only persistence and assuming that the flow of time is the integers. This simple assumption will allow us to present our point of view of how to evaluate a formula at a point or at an interval. It will also ensure we are still within what is expressible in first-order logic. Compare this with Chapter 13 of [Gabbay *et al.*, 1994].

Assume that the atom q is true at the maximal intervals $[Xx_n, y_n], x_n \leq y_n < x_{n+1}$. Then $\sim q$ is true at the intervals $[y_n + 1, x_{n+1} - 1]$, a sequence of the same form, i.e. $y_n + 1 \leq x_{n+1} - 1$ and $x_{n+1} - 1 < y_{n+1} + 1$.

It is easy to compute the intervals corresponding to the truth values of conjunctions: we take the intersection:

If $I_j = \bigcup_n [x_n^j, y_n^j]$ then $I_1 \cap I_2 = \bigcup_n [x_n, y_n]$ and the points x_n, y_n can be effectively linearly computed. Also, if I_j is the interval set for A_j , the interval set for $U(A_1, A_2)$ can be effectively computed.

In Fig. 8, $U(A_1, A_2)$ is true at $[u_k, y_n - 1], [u_k, y_{n+1} - 1]$ which simplifies to the maximal $[u_k, y_{n+1} - 1]$.

The importance of the above is that we can regard a query formula of the full language with Until and Since as an operator on the model (database)

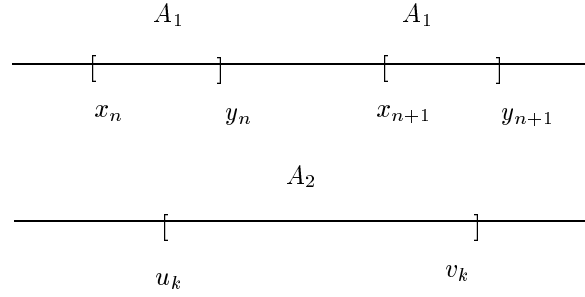


Figure 8.

to give a new database. If the database Δ gives for each atom or its negation the set of intervals where it is true, then a formula A operates on Δ to give the new set of intervals Δ_A ; thus to answer $\Delta?t : A$ the question we ask is $t \in \Delta_A$. The new notion is that the query operates on the model.

This approach was adopted by I. Torsun and K. Manning when implementing the query language USF. The complexity of computation is polynomial (n^2). Note that although we have restricted the database formulae to atoms, we discovered that for no additional cost we can increase the query language to include the connectives Since and Until. As we have seen in Volume 1, in the case of integers the expressive power of Since and Until is equivalent to quantification over time points.

To give the reader another glimpse of what is to come, note that intuitively we have a couple of options:

1. We can assume persistence of atoms and negation of atoms. In this case we can express temporal models in first-order logic. The query language can be full Since and Until logic. This option does not allow for recurrence. In practical terms this means that we cannot generate or easily control recurrent events. Note that the database does not need to contain Horn clauses as data. Clauses of the form $\Box(\text{present wff}_1 \rightarrow \text{present wff}_2)$ are redundant and can be eliminated (this has to be properly proved!). Clauses of the form $\Box(\text{past wff}_1 \rightarrow \text{present wff}_2)$ are not allowed as they correspond to recurrence;
2. This option wants to have recurrence, and is not interested in first-order expressibility. How do we generate recurrence?

The language USF (which was introduced for completely different reasons) allows one to generate the database using rules of the form $\Box(\text{past formula} \rightarrow \text{present or future formula})$.

The above rules, together with some initial items of data of the form $t : A$, A a literal, can generate persistent and recurrent models.

7 NATURAL NUMBERS TIME LOGICS

Certainly the most studied and widely used temporal logic is one based on a natural numbers model of time. The idea of discrete steps of time heading off into an unbounded future is perfect for many applications in computer science. The natural numbers also form quite a well-known structure and so a wide-range of alternative techniques can be brought to bear. Here we will have a brief look at this temporal logic, PTL, at another, stronger natural numbers time temporal logic *USF* and at the powerful automata technique which can be used to help reason in and about such logics.

7.1 PTL

PTL is a propositional temporal logic with semantics defined on the natural numbers time. It does not have past time temporal connectives because, as we will see, they are not strictly necessary, and, anyway, properties of programs (or systems or machines) are usually described in terms of what happens in the future of the start. So PTL is somewhat, but not exactly, like the propositional logic of the until connective over natural numbers time. By the way, PTL stands for Propositional Temporal Logic because computer scientists are not very interested in any of the other propositional temporal logics which we have met. PTL is also sometimes known as PLTL, for Propositional Linear Temporal Logic, because the only other propositional temporal logic of even vague interest to computer scientists is one based on branching time.

PTL does have a version of the until connective and also a next-time (or tomorrow) connective. The until connective, although commonly written as U , is not the same as the connective U which we have met before. Also, and much less importantly, it is usually written in an infix manner as in pUq . In PTL we write pUq iff either q is true now or q is true at some time in the future and we have p true at all points between now and then, including now but not including then. This is called a non-strict version of U : hence here we will use the notation U^{ns} . From the beginning in [Pnueli, 1977], temporal logic work with computer science applications in mind has used the non-strict version of until. In much of the work on temporal logic with a philosophical or linguistic bent, and in this chapter, $U(q, p)$ means that q is true in the future and p holds at all points strictly in between. This is called the strict version of until.

To be more formal let us here define a structure as a triple $(\mathbb{N}, <, h)$ where $h : L \rightarrow \wp(\mathbb{N})$ is the valuation of the atoms from L . In effect, we have

an ω -long sequence $\sigma = \langle \sigma_0, \sigma_1, \dots \rangle$ of states (i.e. subsets of L) with each $\sigma_i = \{p \in L \mid i \in h(p)\}$. Many presentations of PTL use this sequence of states notation. The truth of a PTL formula A at a point n of a structure $\mathcal{T}$, written $\mathcal{T}, n \models A$ is defined by induction on the construction of A as usual. The clauses for atoms and Booleans are as usual. Define:

$$\begin{aligned} \mathcal{T}, n \models XA \text{ iff } & \mathcal{T}, n+1 \models A, \text{ and} \\ \mathcal{T}, n \models AU^{ns}B \text{ iff } & \text{there is some } m \geq n \text{ such that } \mathcal{T}, m \models B \text{ and} \\ & \text{for all } j, \text{ if } n \leq j < m \text{ then } \mathcal{T}, j \models A. \end{aligned}$$

Note that in PTL, there are also non-strict versions of all the usual abbreviations. For example $FA = \top U^{ns}A$ holds iff A holds now or at some time in the future, and $GA = \neg F\neg A$ holds iff A holds now and at all time in the future. In many presentations of PTL, the symbols $\Diamond$ and $\Box$ are used instead of F and G .

In the case of natural numbers time (as in PTL) it is easy to show that the language with strict U is equally expressive as the language with both the next operator X and non-strict U^{ns} . We can define a meaning preserving translation π which preserves atoms and respects Boolean connectives via:

$$\pi U(A, B) = X(\pi(B)U^{ns}\pi(A)).$$

Similarly we can define a meaning preserving translation ρ which preserves atoms and respects Boolean connectives via:

$$\begin{aligned} \rho XA &= U(A, p_0 \wedge \neg p_0), \\ \rho(BU^{ns}A) &= \rho(A) \vee (\rho(B) \wedge U(\rho(A), \rho(B))). \end{aligned}$$

It is easy to show that these are indeed meaning preserving. Notice, though, that is computationally expensive to translate between the non-strict until and strict until (using ρ) as there may be an exponential blow-up in formula length.

In some earlier presentations of PTL, the definitions of the concepts of satisfaction and satisfiability are different from ours. In typical PTL applications, it makes sense to concentrate almost exclusively on the truth of formulas in structures when evaluated at time 0. Thus we might say that structure $(\mathbb{N}, <, h)$ satisfies A iff $(\mathbb{N}, <, h), 0 \models A$. Other presentations define satisfaction by saying $(\mathbb{N}, <, h)$ satisfies A iff for all n , $(\mathbb{N}, <, h), n \models A$. Recall that in this chapter we actually say that $(\mathbb{N}, <, h)$ satisfies A iff there is some n such that $(\mathbb{N}, <, h), n \models A$. In general it is said that a formula A is satisfiable iff there is some structure which satisfies A (whatever notion of satisfaction is used). We will not explore the subtle details of what is sometimes known as the *anchored* versus *floating* version of temporal logics, except to say that no difficult or important issues of expressiveness or axiomatization etc, are raised by these differences. See [Manna and Pnueli, 1988] for more on this.

Given the equivalence of PTL and the temporal logic with U over natural numbers time and our separation result for the $L(U, S)$ logic over the natural numbers, it is easy to see why S (and any other past connective) is not needed. Suppose that we want to check the truth of a formula A in the $L(U, S)$ language at time 0 in a structure $\mathcal{T} = (\mathbb{N}, <, h)$. If we separate A into a Boolean combination of syntactically pure formulas we will see that, for each of the pure past formulas C , either C evaluates at time 0 to true in every structure or C evaluates at time 0 to false in every structure. Thus, we can effectively find some formula B in the language with U such that evaluating A at time 0 is equivalent to evaluating B at time 0. We know that B is equivalent to some formula of PTL which we can also find effectively. Thus adding S adds no expressiveness to PTL (in this specific sense). However, some of the steps in eliminating S may be time-consuming and it may be natural to express some useful properties with S . Thus there are motivations for introducing past-time connectives into PTL. This is the idea seen in [Lichtenstein *et al.*, 1985].

7.2 An axiomatization of PTL

We could axiomatize PTL using either the IRR rule or the techniques of [Reynolds, 1992]. However, the first axiomatization for PTL, given in [Gabbay *et al.*, 1980], uses a different approach with an interesting use of something like the computing concept of fairness. The axioms and proof in [Gabbay *et al.*, 1980] were actually given for a strict version of the logic but, as noted in [Gabbay *et al.*, 1980], it is easy to modify it for the non-strict version, what has now become the official PTL.

The inference rules are modus ponens and generalization,

$$\frac{A, A \rightarrow B}{B} \quad \frac{A}{G^{ns}A} \quad .$$

The axioms are all substitution instances of the following:

- (1) all classical tautologies,
- (2) $G^{ns}(A \rightarrow B) \rightarrow (G^{ns}A \rightarrow G^{ns}B)$
- (3) $X\neg A \rightarrow \neg XA$
- (4) $X(A \rightarrow B) \rightarrow (XA \rightarrow XB)$
- (5) $G^{ns}A \rightarrow A \wedge XG^{ns}A$
- (6) $G^{ns}(A \rightarrow XA) \rightarrow (A \rightarrow G^{ns}A)$
- (7) $AU^{ns}B \rightarrow F^{ns}A$
- (8) $AU^{ns}B \leftrightarrow (B \vee (A \wedge X(AU^{ns}B)))$

The straightforward induction on the lengths of proof gives us the soundness result. The completeness result which is really a *weak completeness* result — the logic is not compact — follows.

THEOREM 95. *If A is valid in PTL then $\vdash A$ (A is a theorem of the axiom system).*

Proof. We give a sketch. The details are left to the reader: or see [Gabbay *et al.*, 1980] (but note the use of strict versions of connectives in that proof). It is enough to show that if A is consistent then A is satisfiable.

We use the common Henkin technique of forming a model of a consistent formula in a modal logic out of the maximal consistent sets of formulae. These are the infinite sets of formulae which are each maximal in not containing some finite subsets whose conjunction is inconsistent. In our case this model will not have the natural numbers flow of time but will be a more general, not necessarily linear structure with a more general definition of truth for the temporal connectives.

Let $\mathcal{C}$ contain all the maximally consistent sets of formulae. This is a non-linear model of A with truth for the connectives defined via the following (accessibility) relations: for each $\Gamma, \Delta \in \mathcal{C}$, say $\Gamma R_+ \Delta$ iff $\{B \mid XB \in \Gamma\} \subseteq \Delta$ and $\Gamma R_< \Delta$ iff $\{B \mid XG^{ns} B \in \Gamma\} \subseteq \Delta$. For example, if we call this model M then for each $\Gamma \in \mathcal{C}$, we define $M, \Gamma \models p$ iff $p \in \Gamma$ for any atom p and $M, \Gamma \models XB$ iff there is some $\Delta \in \mathcal{C}$, such that $\Gamma R_+ \Delta$ and $M, \Delta \models B$. The truth of formulas of the form $B_1 U^{ns} B_2$ is defined via paths through $\mathcal{C}$ in a straightforward way.

The Lindenbaum technique shows us that there is some $\Gamma_0 \in \mathcal{C}$ with $A \in \Gamma_0$. Using this and the fact that $R_<$ is the transitive closure of R_+ , we can indeed show that $M, \Gamma_0 \models A$.

There is also a common technique for taking this model and factoring out by an equivalence relation to form a finite but also non-linear model. This is the method of *filtration*. See [Gabbay *et al.*, 1994].

To do this in our logic, we first limit ourselves to a finite set of interesting formulae:

$$\text{cl}(A) = \{B, \neg B, XB, X\neg B, F^{ns} B, F^{ns} \neg B \mid B \text{ is a subformula of } A\}.$$

Now we define $C = \{\Gamma \cap \text{cl}(A) \mid \Gamma \in \mathcal{C}\}$ and we impose a relation R_X on C via $a R_X b$ iff there exist $\Gamma, \Delta \in \mathcal{C}$ such that $a = \Gamma \cap \text{cl}(A)$, $b = \Delta \cap \text{cl}(A)$ and $\Gamma R_+ \Delta$.

To build natural numbers flowed model of A we next find an ω -sequence σ of sets from C starting at $\Gamma_0 \cap \text{cl}(A)$ and proceeding via the R_X relation in such a way that if the set Γ appears infinitely often in the sequence then each of its R_X -successors do too. This might be called a *fair* sequence. We can turn σ into a structure $(\mathbb{N}, <, h)$ via $i \in h(p)$ iff $p \in \sigma_i$ (for all atoms p). This is enough to give us a truth lemma by induction on all formulae $B \in \text{cl}(A)$: namely, $B \in \sigma_i$ iff $(\mathbb{N}, <, h), i \models B$. Immediately we have $(\mathbb{N}, <, h), 0 \models A$ as required. ■

7.3 *S1S and Fixed point languages*

PTL is expressively complete in respect of formulas evaluated at time 0. That is, for any first-order monadic formula $\phi(P_1, \dots, P_n, x)$ in the language with $<$ with one free variable, there is a PTL formula A such that for any structure $(\mathbb{N}, <, h)$,

$$(\mathbb{N}, <, h), 0 \models A \text{ iff } (\mathbb{N}, <) \models \phi(h(p_1), \dots, h(p_n), 0).$$

To see this just use the expressive completeness of the $L(U, S)$ logic over natural numbers time, separation, the falsity of since at zero, and the translation to PTL.

However, there are natural properties which can not be expressed. For example, there is no PTL formula which is true exactly at the even numbers (see [Wolper, 1983] for details) and we can not say that property p holds at each even-numbered time point. The reason for this lack of expressivity in an expressively complete language is simply that there are natural properties like evenness which can not be expressed in the first-order monadic theory of the natural numbers.

For these reasons there have been many and varied attempts to increase the expressiveness of temporal languages over the natural numbers. There has also been a need to raise a new standard of expressiveness. Instead of comparing languages to the first-order monadic theory of the natural numbers, languages are compared to another traditional second-order logic, the full second-order logic of one successor function, commonly known as *S1S*.

There are several slightly different ways of defining *S1S*. We can regard it as an ordinary first-order logic interpreted in a structure which actually consists of sets of natural numbers. The signature contains the 2-ary subset relation $\subseteq$ and a 2-ary ordering relation symbol *succ*. Subset is interpreted in the natural way while *succ*(A, B) holds for sets A and B iff $A = \{n\}$ and $B = \{n+1\}$ for some number n . To deal with a temporal structure using atoms from L we also allow the symbols in L as constant symbols in the language: given an ω -structure σ , the interpretation of the atom p is just the set of times at which p holds.

Consider the example of the formula

$$J(a, b, z)(b \subseteq z) \wedge \forall uv(succ(u, v) \wedge (v \subseteq z) \wedge (u \subseteq a) \rightarrow (u \subseteq z))$$

with constants a, b and z . This will be true of a natural numbers flowed temporal structure with atoms a, b and z iff at every time at which $aU^{ns}b$ holds, we also have z holding.

In fact, it is straightforward to show that the language of *S1S* is exactly as expressive as the full second-order monadic language of the natural number order. Thus it is more expressive than the first-order monadic language. A well-known and straightforward translation gives an *S1S* version of any

temporal formula. We can translate any temporal formula α using atoms from L into an $S1S$ formula $(*\alpha)(x)$ with a free variable x :

$$\begin{aligned}
*p &= (x = p) \\
(\neg\alpha) &= \neg(\alpha) \\
*(\alpha \wedge \beta) &= *\alpha \wedge *\beta \\
(X\alpha) &= \forall y((\alpha)(y) \rightarrow \forall uv(succ(u, v) \wedge (u \subseteq x) \rightarrow (v \subseteq y))) \\
(\alpha U^{ns} \beta) &= \forall ab((\alpha)(a) \wedge (*\beta)(b)) \rightarrow \\
&\quad (J(a, b, x) \wedge ((\forall y(J(a, b, y) \rightarrow (x \subseteq y)))) \\
\text{where } J(a, b, z) &= (b \subseteq z) \\
&\quad \wedge \forall uv(succ(u, v) \wedge (v \subseteq z) \wedge (u \subseteq a) \rightarrow (u \subseteq z))
\end{aligned}$$

An easy induction (on the construction of α) shows that $\sigma \models (*\alpha)(S)$ iff S is the set of times at which α holds.

In order to reach the expressiveness of $S1S$, temporal logics are often given an extra second-order capability involving some kind of quantification over propositions. See, for example, ETL [Wolper, 1983] and the quantified logics in chapter 8 of [Gabbay *et al.*, 1994]. One of the most computationally convenient ways of adding quantification is via the introduction of fixed-point operators into the language. This has been done in [Banniegbal and Barringer, 1986] and in [Gabbay, 1989]. We briefly look at the example of USF from [Gabbay, 1989].

In chapter 8 of [Gabbay *et al.*, 1994] it is established that $S1S$ is exactly as expressive as USF . In order to use the automata-based decision procedures (which we meet in subsection 7.5 below) to give us decision procedures about temporal logics we need only know that the temporal logic of interest is less expressive than USF (or equivalently $S1S$) — as they mostly are. So it is worth here briefly recalling the definition of USF .

In fact we start with the very similar language UYF . We often write $\bar{x}, \bar{a}, \dots$, for *tuples* — finite sequences of variables, atoms, elements of a structure, etc. If $S \subseteq \mathbb{N}$, we write $S + 1$ (or $1 + S$) for $\{s + 1 \mid s \in S\}$.

We start by developing the syntax and semantics of the fixed point operator. This is not entirely a trivial task. We will fix an infinite set of propositional atoms, with which our formulae will be written; we write $p, q, r, s, \dots$ for atoms.

DEFINITION 96.

1. The set of formulae of UYF is the smallest class closed under the following:
 - (a) Any atom q is a formula of UYF , as is $\top$.
 - (b) If A is a formula so is $\neg A$.

- (c) If A is a formula so is YA . We read Y as ‘yesterday’.
 - (d) If A and B are formulae, so are $A \wedge B$ and $U(A, B)$. ($A \vee B$ and $A \rightarrow B$ are regarded as abbreviations.)
 - (e) Suppose that A is a formula such that every occurrence of the atom q in A not within the scope of a φq is within the scope of a Y but not within the scope of a U . Then $\varphi q A$ is a formula. (The conditions ensure that $\varphi q A$ has fixed point semantics.)
2. The *depth of nesting* of φ s in a formula A is defined by induction on its formation: formulae formed by clause (a) have depth 0, clause (e) adds 1 to the depth of nesting, clauses (b) and (c) leave it unchanged, and in clause (d), the depth of nesting of $U(A, B)$ and $A \wedge B$ is the maximum of the depths of nesting of A and B . So, for example, $\neg \varphi r (\neg Y r \wedge \varphi q Y (q \rightarrow r))$ has depth of nesting of 2.
3. A UYF-formula is said to be a YF-formula if it does not involve U .
4. Let A be a formula and q an atom. A *bound occurrence* of q in A is one in a subformula of A of the form $\varphi q B$. All other occurrences of q in A are said to be *free*. An occurrence of q in A is said to be *pure past* in A if it is in a subformula of A of the form YB but not in a subformula of the form $U(B, C)$. So $\varphi q A$ is well-formed if and only if all free occurrences of q in A are pure past.

An *assignment* is a map h providing a subset $h(q)$ of $\mathbb{N}$ for each atom q . If h, h' are assignments, and $\bar{q}$ a tuple of atoms, we write $h =_{\bar{q}} h'$ if $h(r) = h'(r)$ for all atoms r not occurring in $\bar{q}$. If $S \subseteq \mathbb{N}$ and q is an atom, we write $h_{q/S}$ for the unique assignment h' satisfying: $h' =_q h$, $h'(q) = S$.

For each assignment h and formula A of UYF we will define a subset $h(A)$ of $\mathbb{N}$, the interpretation of A in $\mathbb{N}$. Intuitively, $h(A) = \{n \in \mathbb{N} \mid A \text{ is true at } n \text{ under } h\} = \{n \in \mathbb{N} \mid (\mathbb{N}, <, h), n \models A\}$. We will ensure that, whenever $\varphi q A$ is well-formed,

$$(*) \quad h(\varphi q A) \text{ is the unique } S \subseteq \mathbb{N} \text{ such that } S = h_{q/S}(A).$$

DEFINITION 97. We define the semantics of UYF by induction. Let h be an assignment. If A is atomic then $h(A)$ is already defined. We set:

- $h(\top) = \mathbb{N}$.
- $h(\neg A) = \mathbb{N} \setminus h(A)$.
- $h(YA) = h(A) + 1$.
- $h(A \wedge B) = h(A) \cap h(B)$.
- $h(U(A, B)) = \{n \in \mathbb{N} \mid \exists m > n (m \in h(A) \wedge \forall m' (n < m' < m \rightarrow m' \in h(B)))\}$.

- Finally, if φqA is well-formed we define $h(\varphi qA)$ as follows. First define assignments h_n ($n \in \mathbb{N}$) by induction: $h_0 = h, h_{n+1} = (h_n)_{q/h_n(A)}$. Then $h(\varphi qA) \stackrel{\text{def}}{=} \{n \in \mathbb{N} \mid n \in h_n(A)\} = \{n \in \mathbb{N} \mid n \in h_{n+1}(q)\}$.

To establish (*) we need a theorem.

THEOREM 98 (fixed point theorem).

1. Suppose that A is any UYF-formula and φqA is well formed. Then if h is any assignment, there is a unique subset $S = h(\varphi qA)$ of $\mathbb{N}$ such that $S = h_{q/S}(A)$. Thus, regarding $S \mapsto h_{q/S}(A)$ as a map $\alpha : \wp(\mathbb{N}) \rightarrow \wp(\mathbb{N})$ (depending on h, A), α has a unique fixed point $S \subseteq \mathbb{N}$, and we have $S = h(\varphi qA)$. For any h , $h(A) = h(q) \iff h(\varphi qA) = h(q)$.
2. If q has no free occurrence in a formula A and $g =_q h$, then $g(A) = h(A)$.
3. If φqA is well-formed and r is an atom not occurring in A , then for all assignments h , $h(\varphi qA) = h(\varphi rA(q/r))$, where $A(q/r)$ denotes substitution by r for all free occurrences of q in A .

We define USF using the first-order connectives Until and Since as well as the fixed point operator. The logic UYF is just as expressive as USF: Yq is definable in USF by the formula $S(q, \perp)$, while $S(p, q)$ is definable in UYF by $\varphi rY(p \vee (q \wedge r))$. Using UYF allows easier proofs and stronger results.

As an example, consider the formula

$$A = \varphi q(\neg Yq).$$

It is easy to see that A holds in a structure iff q holds exactly at the even numbered times.

7.4 Decision Procedures

There are many uses for PTL (and extensions such as USF) in describing and verifying systems. Once again, an important task required in many of these applications is determining the validity (or equivalently satisfiability) of formulas. Because it is a widely-used logic and because the natural numbers afford a wide-variety of techniques of analysis, there are several quite different ways of approaching decision procedures here. The main avenues are via finite model properties, tableaux, automata and resolution techniques.

The first proof of the decidability of PTL was based on automata. The pioneer in the development of automata for use with infinite linear structures is Büchi in [1962]. He was interested in proving the decidability of $S1S$ as

a very restricted version of second-order arithmetic. We will look at his proof briefly in subsection 7.5 below. By the time that temporal logic was being introduced to computer scientists in [Pnueli, 1977], it was well known (via [Kamp, 1968b]) that temporal logic formulae can be expressed in the appropriate second-order logic and so via *S1S* we had the first decision procedure for PTL (and *USF*).

Unfortunately, deciding *S1S* is non-elementarily complex (see [Robertson, 1974]) and so this is not an efficient way to decide PTL. Tableaux [Wolper, 1983; Lichtenstein *et al.*, 1985; Emerson, 1990], resolution [Fisher, 1997] and automata approaches (which we meet in subsection 7.5 below) can be much more efficient.

The first PSPACE algorithm for deciding PTL was given in [Sistla and Clarke, 1985]. This result uses a finite model property. We show that if a PTL formula A of length n is satisfiable over the natural numbers then it is also satisfiable in a non-linear model of size bounded by a certain exponential in n . The model has a linear part followed by a loop. It is a straightforward matter to guess the truths of atoms at the states on this structure and then check the truth of the formula. the guessing and checking can be done “on the fly”, i.e. simultaneously as we move along the structure in such a way that we do not need to store the whole structure. So we have an NPSPACE algorithm which by the well-known result in [Savitch, 1970] can give us a PSPACE one.

In [Sistla and Clarke, 1985] it was also shown that deciding PTL is PSPACE-hard. This is done by encoding the running of any polynomial space bounded Turing machine into the logic.

For complexities of deciding the logic with strict *U* and the *USF* version see chapter 15 of [Gabbay *et al.*, 1994]. They are both PSPACE-complete. The same finite model property ideas are used.

7.5 Automata

Automata are finite state machines which are very promising objects to help with deciding the validity of temporal formulae. In some senses they are like formulae: they are finite objects and they distinguish some temporal structures—the ones which they accept—from other temporal structures in much the same way that formulae are true (at some point) in some structures are not in others. In other senses automata are like structures: they contain states and relate each state with some successor states. Being thus mid-way between formulae and structures allows automata to be used to answer questions—such as validity—about the relation between formulae and structures.

An automaton is called *empty* iff it accepts no structures and it turns out to be relatively easy to decide whether a given automaton is empty or not. This is surprising because empty automata can look quite complicated

in much the same way as unsatisfiable formulae can. This fact immediately suggests a possible decision procedure for temporal formulae. Given a formula we might be able to find an automaton which accepts exactly the structures which are models of the formula. If we now test the automaton for emptiness then we are effectively testing the formula for unsatisfiability. Validity of a formula corresponds to emptiness of an automaton equivalent to the negation of the formula.

This is the essence of the incredibly productive automata approach to theorem proving. We only look in detail at the case of PTL on natural numbers time.

The idea of (finite state) automata developed from pioneering attempts by Turing to formalize computation and by Kleene [1956] to model human psychology. The early work (see, for example, [Rabin and Scott, 1959]) was on finite state machines which recognized finite words. Such automata have provided a formal basis for many applications from text processing and biology to the analysis of concurrency. There has also been much mathematical development of the field. See [Perrin, 1990] for a survey.

The pioneer in the development of automata for use with infinite linear structures is Büchi in [1962] in proving the decidability of $S1S$. This gives one albeit inefficient decision procedure for PTL. There are now several useful ways of using the automata stepping stone for deciding the validity of PTL formulae.

The general idea is to translate the temporal formula into an automaton which accepts exactly the models of the formula and then to check for emptiness of the automaton. Variations arise when we consider that there are several different types of automata which we could use and that the translation from the formula can be done in a variety of ways.

Let us look at the automata first. For historical reasons we will switch now to a language Σ of letters rather than keep using a language of propositional atoms. The nodes of trees will be labelled by a single letter from Σ . In order to apply the results in this section we will later have to take the alphabet Σ to be 2^P where P is the set of atomic propositions.

A Σ (linear) Büchi automaton is a 4-tuple $A = (S, T, S_0, F)$ where

- S is a finite non-empty set called the set of *states*,
- $T \subseteq S \times \Sigma \times S$ is the *transition table*,
- $S_0 \subseteq S$ is the *initial state set* and
- $F \subseteq S$ is the *set of accepting states*.

A *run* of A on an ω -structure σ is a sequence of states $\langle s_0, s_1, s_2, \dots \rangle$ from S such that $s_0 \in S_0$ and for each $i < \omega$, $(s_i, \sigma_i, s_{i+1}) \in T$. We assume that automata never grind to a halt: i.e. we assume that for all $s \in S$, for all $a \in \Sigma$, there is some $s' \in S$ such that $(s, a, s') \in T$.

We say that the automaton accepts σ iff there is a run $\langle s_0, s_1, \dots \rangle$ such that $s_i \in F$ for infinitely many i .

One of the most useful results about Büchi automata, is that we can complement them. That is given a Büchi automata A reading from the language Σ we can always find another Σ Büchi automata $\bar{A}$ which accepts exactly the ω -sequences which A rejects. This was first shown by Büchi in [1962] and was an important step on the way to his proof of the decidability of $S1S$. The automaton $\bar{A}$ produced by Büchi's method is double exponential in the size of A but more recent work in [Sistla *et al.*, 1987] shows that complementation of Büchi automata can always be singly exponential.

As we will see below, it is easy to complement an automaton if we can find a deterministic equivalent. This means an automaton with a unique initial state and a transition table $T \subseteq S \times \Sigma \times S$ which satisfies the property that for all $s \in S$, for all $a \in \Sigma$, there is a unique $s' \in S$ such that $(s, a, s') \in T$. A deterministic automaton will have a unique run on any given structure.

Two automata are equivalent iff they accept exactly the same structures.

The problem with Büchi automata is that it is not always possible to find a deterministic equivalent. A very short argument, see example 4.2 in [Thomas, 1990], shows that the non-deterministic $\{a, b\}$ automaton which recognizes exactly the set

$$L = \{\sigma | a \text{ appears only a finite number of times in } \sigma\}$$

can have no deterministic equivalent.

One of our important tasks is to decide whether a given automaton is empty i.e. accepts no ω -structures. For Büchi automata this can be done in linear time [Emerson and Lei, 1985] and co-NLOGSPACE [Vardi and Wolper, 1994].

The lack of a determinization result for Büchi automata led to a search for a class of automata which is as expressive as the class of Büchi automata but which is closed under finding deterministic equivalents. Muller automata were introduced by Muller in [Muller, 1963] and in [Rabin, 1972] variants, now called Rabin automata, were introduced.

The difference is that the accepting condition can require that certain states do not come up infinitely often. There are several equivalent ways of formalizing this. The Rabin method is, for a Σ -automata with state set S , to use a set $\mathcal{F}$, called the set of *accepting pairs*, of pairs of sets of states from S , i.e. $\mathcal{F} \subseteq \wp(S) \times \wp(S)$.

We say that the Rabin automaton $A = (S, S_0, T, \mathcal{F})$ accepts σ iff there is some run $\langle s_0, s_1, s_2, \dots \rangle$ (as defined for Büchi automata) and some pair $(U, V) \in \mathcal{F}$ such that no state in V is visited infinitely often but there is some state in U visited infinitely often.

In fact, Rabin automata add no expressive power compared to Büchi automata, i.e. for every Rabin automaton there is an equivalent Büchi automaton. The translation [Choueka, 1974] is straightforward and, as it

essentially just involves two copies of the Rabin automata in series with a once-only non-deterministic transition from the first to the second, it can be done in polynomial time. The converse equivalence is obvious.

The most important property of the class of Rabin automata is that it is closed under determinization. In [1966], McNaughton, showed that any Büchi automaton has a deterministic Rabin equivalent. There are useful accounts of McNaughton's theorem in [Thomas, 1990] and [Hodkinson, 200]. McNaughton's construction is doubly exponential. It follows from McNaughton's result that we can find a deterministic equivalent of any Rabin automaton: simply first find a Büchi equivalent and then use the theorem.

The determinization result gives us an easy complementation result for Rabin automata: given a Rabin automata we can without loss of generality assume it is deterministic and complementing a deterministic automaton is just a straightforward negation of the acceptance criteria.

To decide whether Rabin automata are empty can be done with almost the same procedure we used for Büchi case. Alternatively, one can determinize the automaton A , and translate the deterministic equivalent into a deterministic Rabin automaton A' recognizing ω -sequences from the one symbol alphabet $\{a_0\}$ such that A' accepts some sequence iff A does. It is very easy to tell if A' is empty.

Translating formulae into Automata

The first step in using automata to decide a temporal formula is to translate the temporal formula into an equivalent automata: i.e. one that accepts exactly the models of the formula. There are direct ways of making this translation, e.g., in [Sherman *et al.*, 1984] (via a nonelementarily complex procedure). However, it is easier to understand some of the methods which use a stepping stone in the translation: $S1S$.

We have seen that the translation from PTL into $S1S$ is easy. The translation of $S1S$ into an automaton is also easy, given McNaughton's result: it is via a simple induction. Suppose that the $S1S$ sentence uses constants from the finite set P . We proceed by induction on the construction of the sentence. The automaton for $p \subseteq q$ simply keeps checking that $p \rightarrow q$ is true of the current state and falls into a fail state sink if not. The other base cases, of $p = q$ and $succ(p, q)$ are just as easy. Conjunction requires a standard construction of conjoining automata using the product of the state sets. Negation can be done using McNaughton's result to determinize the automaton for the negated subformula. It is easy to find the complement of a deterministic automaton. The case of an existential quantification, e.g., $\exists y \rho(y)$, is done by simply using non-determinism to guess the truth of the quantified variable at each step.

Putting together the results above gives us several alternative approaches to deciding validity of PTL formulae. One route is to translate the formula

into $S1S$, translate the $S1S$ formula into a Büchi automaton as above and then check whether that is empty.

The quickest route is via the alternating automata idea of [Brzozowski and Leiss, 1980] — a clever variation on the automata idea. By translating a formula into one of these automata, and then using a guess and check on the fly procedure, we need only check a polynomial number of states (in the size of ϕ) and then (nondeterministically) move on to another such small group of states. This gives us a PSPACE algorithm. From the results of [Sistla and Clarke, 1985], we know this is best possible as a decision procedure.

Other Uses of Automata

The decision algorithm above using the translation into the language $S1S$ can be readily extended to allow for past operators or fixed point operators or both to appear in the language. This is because formulae using these operators can be expressed in $S1S$.

Automata do not seem well suited to reasoning about dense time or general linear orders. However, the same strategy as we used for PTL also works for the decidability of branching time logics such as CTL*. The only difference is that we must use tree automata. These were invented by Rabin in his powerful results showing the decidability of $S2S$, the second-order logic of two successors. See [Gurevich, 1985] for a nice introduction.

8 EXECUTABLE TEMPORAL LOGIC

Here we describe a useful paradigm in executable logic: that of the *declarative past and imperative future*. A future statement of temporal logic can be understood in two ways: the declarative way, that of describing the future as a temporal extension; and the imperative way, that of making sure that the future will happen the way we want. Since the future has not yet happened, we have a language which can be both declarative and imperative. We regard our theme as a natural meeting between the imperative and declarative paradigms.

More specifically, we describe a temporal logic with Since, Until and fixed point operators. The logic is based on the natural numbers as the flow of time and can be used for the specification and control of process behaviour in time. A specification formula of this logic can be automatically rewritten into an executable form. In an executable form it can be used as a program for controlling process behaviour. The executable form has the structure ‘If A holds in the past then do B ’. This structure shows that declarative and imperative programming can be integrated in a natural way.

Let $\mathcal{E}$ be an environment in which a program $\mathcal{P}$ is operating. The exact nature of the environment or the code and language of the program are not

immediately relevant for our purposes. Suppose we make periodic checks at time $t_0, t_1, t_2, t_3, \dots$ on what is going on in the environment and what the program is doing. These checks could be made after every unit execution of the program or at some key times. The time is not important to our discussion. What is important is that we check at each time the truth values of some propositions describing features of the environment and the program. We shall denote these propositions by $a_1, \dots, a_m, b_1, \dots, b_k$. These propositions, which we regard as units taking truth values $\top$ (true) or $\perp$ (false) at every checkpoint, need not be expressible in the language of the program $\mathcal{P}$, nor in the language used to describe the environment. The program may, however, in its course of execution, change the truth values of some of the propositions. Other propositions may be controlled only by the environment. Thus we assume that $a_1, \dots, a_m$ are capable of being influenced by the program while $b_1, \dots, b_k$ are influenced by the environment. We also assume that when at checktime t_n we want the program to be executed in such a way as to make the proposition a_i true, then it is possible to do so. We express this command by writing **exec** (a_i). For example, a_1 can be ‘print the screen’ and b_1 can be ‘there is a read request from outside’; a_1 can be controlled by the program while b_1 cannot. **exec** (a_1) will make a_1 true.

To illustrate our idea further, we take one temporal sentence of the form

$$(1) \quad G[\bullet a \Rightarrow Xb].$$

$\bullet$ is the ‘yesterday’ operator, X is the ‘tomorrow’ operator, and G is the ‘always in the future’ operator. One can view (1) as a wff of temporal logic which can be either true or false in a temporal model. One can use a temporal axiom system to check whether it is a temporal theorem etc. In other words, we treat it as a formula of logic.

There is another way of looking at it. Suppose we are at time n . In a real ticking forward temporal system, time $n + 1$ (assume that time is measured in days) has not happened yet. We can find the truth value of $\bullet a$ by checking the past. We do not know yet the value of Xb because tomorrow has not yet come. Checking the truth value of $\bullet a$ is a declarative reading of $\bullet a$. However, we need not read Xb declaratively. We do not need to wait and see what happens to b tomorrow. Since tomorrow has not yet come, we can make b true tomorrow if we want to, and are able to. We are thus reading Xb imperatively: ‘make b true tomorrow’.

If we are committed to maintaining the truth of the specification (1) throughout time, then we can read (1) as: ‘at any time t , if $\bullet a$ holds then execute Xb ’, or schematically, ‘if declarative past then imperative future’. This is no different from the Pascal statement

```
if x<5 then x:=x+1
```

In our case we involve whole formulae of logic.

The above is our basic theme. This section makes it more precise. The rest of this introduction sets the scene for it, and the conclusion will describe existing implementations. Let us now give several examples:

EXAMPLE 99 (Simplified Payroll). Mrs Smith is running a babysitter service. She has a list of reliable teenagers who can take on a babysitting job. A customer interested in a babysitter would call Mrs Smith and give the date on which the babysitter is needed. Mrs Smith calls a teenager employee of hers and arranges for the job. She may need to call several of her teenagers until she finds one who accepts. The customer pays Mrs Smith and Mrs Smith pays the teenager. The rate is £10 per night unless the job requires overtime (after midnight) in which case it jumps to £15.

Mrs Smith uses a program to handle her business. The predicates involved are the following:

$A(x)$	x is asked to babysit
$B(x)$	x does a babysitting job
$M(x)$	x works after midnight
$P(x, y)$	x is paid y pounds .

In this set-up, $B(x)$ and $M(x)$ are controlled mainly by the environment and $A(x)$ and $P(x, y)$ are controlled by the program.

We get a temporal model by recording the history of what happens with the above predicates. Mrs Smith laid out the following (partial) specification:

1. Babysitters are not allowed to take jobs three nights in a row, or two nights in a row if the first night involved overtime.
2. Priority in calling is given to babysitters who were not called before as many times as others.
3. Payment should be made the next day after a job is done.

Figure 9 is an example of a partial model of what has happened to a babysitter called Janet. This model may or may not satisfy the specification.

We would like to be able to write down the specification in an intuitive temporal language (or even English) and have it automatically transformed into an executable program, telling us what to do day by day.

EXAMPLE 100 (J. Darlington, L. While [1987]). Consider a simple program $\mathcal{P}$, written in a rewrite language, to merge two queues. There are two merge rules:

R1 $\text{Merge}(a.x, y) = a.\text{merge}(\mathbf{x}, \mathbf{y});$

R2 $\text{Merge}(x, a.y) = a.\text{merge}(\mathbf{x}, \mathbf{y}).$

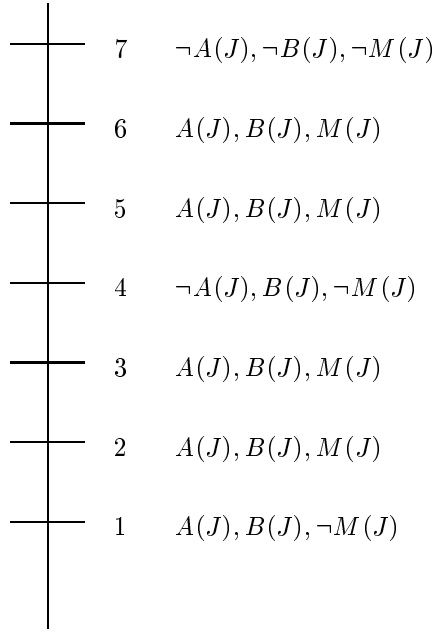


Figure 9. A model for Janet.

That is, we have left or right merges. The environment E with which $\mathcal{P}$ interacts consists of the two queues x and y which get bigger and bigger over time. A real-life example is a policeman merging two queues of traffic. We use $t_0, t_1, t_2, \dots$ as checktimes. The propositions we are interested in are:

- A_1 The program uses the **R1** merge rule (left merge).
- A_2 The program uses the **R2** merge rule (right merge).
- B The left queue is longer than the right queue.

Notice that the proposition B is not under the complete control of the program. The environment supplies the queuing elements, though the merge process does take elements out of the queue. The propositions A_1 and A_2 can be made true by the program, though not in the framework of the rewrite language, since the evaluation is non-deterministic. The program may be modified (or annotated) to a program $\mathcal{P}_1$ which controls the choice of the merge rules. In the general case, for other possible programs and other possible languages, this may not be natural or even possible to do.

EXAMPLE 101 (Loop checking in Prolog). Imagine a Prolog program $\mathcal{P}$ and imagine predicates $A_1, \dots, A_m$ describing at each step of execution which rule is used and what is the current goal and other relevant data. Let B describe the history of the computation. This can be a list of states defined recursively. The loop checking can be done by ensuring that certain temporal properties hold throughout the computation. We can define in this set-up any loop-checking system we desire and change it during execution.

In the above examples, the propositions A_i, B_j change the truth value at each checktime t_k . We thus obtain a natural temporal model for these propositions (see Fig. 10).

	$\vdots$		
3	$a_1 = \perp$	$b_2 = \top$	
2	$a_1 = \perp$	$b_2 = \top$	
1	$a_1 = \top$	$b_2 = \top$	
0	$a_1 = \top$	$b_2 = \perp$	

Figure 10. An example temporal model.

In the above set-up the programmer is interested in influencing the execution of the program within the non-deterministic options available in the programming language. For example, in the merge case one may want to say that if the left queue is longer than the right queue then use the left merge next. In symbols

$$G[B \Rightarrow XA1].$$

In the Prolog case, we may want to specify what the program should do in case of loops, i.e.

$$G[C \wedge PC \Rightarrow D],$$

where C is a complex proposition describing the state of the environment of interest to us (P is the ‘in the past’ operator). $C \wedge PC$ indicate a loop and D says what is to be done. The controls may be very complex and can be made dependent on the data and to change as we go along.

Of course in many cases our additional controls of the execution of $\mathcal{P}$ may be synthesized and annotated in $\mathcal{P}$ to form a new program $\mathcal{P}^*$. There are several reasons why the programmer may not want to do that:

1. The programming language may be such that it is impossible or not natural to synthesize the control in the program. We may lose in clarity and structure.

2. Changes in the control structure may be expensive once the program $\mathcal{P}^*$ is defined and compiled.
3. It may be impossible to switch controlling features on and off during execution, i.e. have the control itself respond to the way the execution flows.
4. A properly defined temporal control module may be applicable as a package to many programming languages. It can give both practical and theoretical advantages.

In this section we follow option 4 above and develop an executable temporal logic for interactive systems. The reader will see that we are developing a logic here that on the one hand can be used for specification (of what we want the program to do) and on the other hand can be used for execution. (How to pass from the specification to the executable part requires some mathematical theorems.) Since logically the two formulations are equivalent, we will be able to use logic and proof theory to prove correctness.

This is what we have in mind:

1. We use a temporal language to specify the desirable behaviour of $\{a_i, b_j\}$ over time. Let $\mathcal{S}$ be the specification as expressed in the temporal language (e.g. $G[b_2 \Rightarrow Xa_1]$).
2. We rewrite automatically $\mathcal{S}$ into $\mathcal{E}$, $\mathcal{E}$ being an executable temporal module. The program $\mathcal{P}$ can communicate with $\mathcal{E}$ at each checktime t_i and get instructions on what to do.

We have to prove that:

- if $\mathcal{P}$ follows the instructions of $\mathcal{E}$ then any execution sequence satisfies $\mathcal{S}$, i.e. the resulting temporal model for $\{a_i, b_j\}$ satisfies the temporal formula $\mathcal{S}$;
- any execution sequence satisfying $\mathcal{S}$ is non-deterministically realizable using $\mathcal{P}$ and $\mathcal{E}$.

The proofs are tough!

Note that our discussion also applies to the case of shared resources. Given a resource to be shared by several processes, the temporal language can specify how to handle concurrent demands by more than one process. This point of view is dual to the previous one. Thus in the merge example, we can view the merge program as a black box which accepts items from two processes (queues), and the specification organizes how the box (program) is to handle that merge. We shall further observe that since the temporal language can serve as a metalanguage for the program $\mathcal{P}$ (controlling its

execution), $\mathcal{P}$ can be completely subsumed in $\mathcal{E}$. Thus the temporal language itself can be used as an imperative language ($\mathcal{E}$) with an equivalent specification element $\mathcal{S}$. Ordinary Horn logic programming can be obtained as a special case of the above.

We can already see the importance of our logic from the following point of view. There are two competing approaches to programming: the declarative one as symbolized in logic programming and Prolog, and the imperative one as symbolized in many well-known languages. There are advantages to each approach and at first impression there seems to be a genuine conflict between the two. The executable temporal logic described in this section shows that these two approaches can truly complement each other in a natural way.

We start with a declarative specification $\mathcal{S}$, which is a formula of temporal logic. $\mathcal{S}$ is transformed into an executable form which is a conjunction of expressions of the form

hold C in the past $\Rightarrow$ **execute** B now.

At any moment of time, the past is given to us as a database; thus **hold**(C) can be evaluated as a goal from this database in a Prolog program. **execute**(B) can be performed imperatively. This creates more data for the database, as the present becomes past. Imperative languages have a little of this feature, e.g. **if** $x < 5$ **let** $x := x + 1$. Here **hold**(C) equates to $x < 5$ and **execute**(B) equates to $x := x + 1$. The $x < 5$ is a very restricted form of a declarative test. On the other hand Prolog itself allows for imperative statements. Prolog clauses can have the form

write(Term) $\Rightarrow$ b

and the goal **write**(Term) is satisfied by printing. In fact, one can accomplish a string of imperative commands just by stringing goals together in a clever way.

We thus see our temporal language as a pointer in the direction of unifying in a logical way the declarative and imperative approaches. The temporal language can be used for planning. If we want to achieve B we try to execute B . The temporal logic will give several ways of satisfying **execute**(B) while **hold**(C) remains true. Any such successful way of executing B is a plan. In logical terms we are looking for a model for our atoms but since we are dealing with a temporal model and the atoms can have an imperative meaning we get a plan. We will try and investigate these points further.

8.1 The logic USF

We describe a temporal system for specification and execution. The logic is USF which we met briefly in section 6 above. It contains the temporal

connectives *since*(S) and *until*(U) together with a fixed point operator φ . The formulae of USF are used for specifying temporal behaviour and these formulae will be syntactically transformed into an executable form. We begin with the definitions of the syntax of USF. There will be four types of well-formed formulae, pure future formulae (talking only about the strict future), pure past formulae (talking only about the strict past), pure present formulae (talking only about the present) and mixed formulae (talking about the entire flow of time).

DEFINITION 102 (Syntax of USF for the propositional case). Let Q be a sufficiently large set of *atoms* (atomic propositions). Let $\wedge, \vee, \neg, \Rightarrow, \top, \perp$ be the usual classical connectives and let U and S be the temporal connectives and φ be the fixed point operator. We define by induction the notions of

- wff (well-formed formula);
- wff^+ (pure future wff);
- wff^- (pure past wff);
- wff^0 (pure present wff).

1. An atomic $q \in Q$ is a pure present wff and a wff. Its atoms are q .
2. Assume A and B are wffs with atoms $\{q_1, \dots, q_n\}$ and $\{r_1, \dots, r_m\}$ respectively. Then $A \wedge B$, $A \vee B$, $A \Rightarrow B$, $U(A, B)$ and $S(A, B)$ are wffs with atoms $\{q_1, \dots, q_n, r_1, \dots, r_m\}$.
 - (a) If both A and B are in $wff^0 \cup wff^+$, then $U(A, B)$ is in wff^+ .
 - (b) If both A and B are in $wff^0 \cup wff^-$, then $S(A, B)$ is in wff^- .
 - (c) If both A and B are in wff^* , then so are $A \wedge B$, $A \vee B$, $A \Rightarrow B$, where wff^* is one of wff^+ , wff^- or wff^0 .
3. $\neg A$ is also a wff and it is of the same type as A with the same atoms as A .
4. $\top$ (truth) and $\perp$ (falsity) are wffs in wff^0 with no atoms.
5. If A is a wff in wff^- (pure past) with atoms $\{q, q_1, \dots, q_n\}$ then $(\varphi q)A$ is a pure past wff (i.e. in wff^-) with the atoms $\{q_1, \dots, q_n\}$.

The intended model for the above propositional temporal language is the set of natural numbers $\mathbb{N} = 0, 1, 2, 3, \dots$ with the ‘smaller than’ relation $<$ and variables $P, Q \subseteq \mathbb{N}$ ranging over subsets. We allow quantification $\forall x \exists y$ over elements of $\mathbb{N}$. So really we are dealing with the monadic language of the model $(\mathbb{N}, <, =, 0, P_i, Q_j \subseteq \mathbb{N})$. We refer to this model also as the *non-negative integers (flow of) time*. A formula of the monadic language will in general have free set variables, and these correspond to the atoms of temporal formulae. See Volume 1 for more details.

DEFINITION 103 (Syntax of USF for the predicate case). Let $Q^* = \{Q_{n_1}^1, Q_{n_2}^2, \dots\}$ be a set of predicate symbols. $Q_{n_i}^i$ is a symbol for an n_i -place predicate. Let $f^* = \{f_{n_1}^1, f_{n_2}^2, \dots\}$ be a set of function symbols. $f_{n_i}^i$ is a function symbol for an n_i -place function. Let $V^* = \{v_1, v_2, \dots\}$ be a set of variables. Let $\wedge, \vee, \neg, \Rightarrow, \top, \perp, \forall, \exists$ be the usual classical connectives and quantifiers and let U and S be the temporal connectives and φ be the fixed point operator. We define by induction the notions of:

- $wff\{x_1, \dots, x_n\}$ wff with free variables $\{x_1, \dots, x_n\}$;
- $wff^+\{x_1, \dots, x_n\}$ pure future wff with the indicated free variables;
- $wff^-\{x_1, \dots, x_n\}$ pure past wff with the indicated free variables;
- $wff^0\{x_1, \dots, x_n\}$ pure present wff with the indicated free variables;
- $term\{x_1, \dots, x_n\}$ term with the indicated free variables.

1. x is a term in $term\{x\}$, where x is a variable.
2. If f is an n -place function symbol and $t_1, \dots, t_n$ are terms with variables $V_1^*, \dots, V_n^* \subseteq V^*$ respectively, then $f(t_1, \dots, t_n)$ is a term with variables $\bigcup_{i=1}^n V_i^*$.
3. If Q is an n -place atomic predicate symbol and we have $t_1, \dots, t_n$ as terms with variables $V_1^*, \dots, V_n^* \subseteq V^*$ respectively, then $Q(t_1, \dots, t_n)$ is an atomic formula with free variables, $\bigcup_{i=1}^n V_i^*$. This formula is pure present as well as a wff.
4. Assume A, B are formulae with free variables $\{x_1, \dots, x_n\}$ and $\{y_1, \dots, y_m\}$ respectively. Then $A \wedge B, A \vee B, A \Rightarrow B, U(A, B)$ and $S(A, B)$ are wffs with the free variables $\{x_1, \dots, x_n, y_1, \dots, y_m\}$.
 - (a) If both A and B are in $wff^0 \cup wff^+$, then $U(A, B)$ is in wff^+ .
 - (b) If both A and B are in $wff^0 \cup wff^-$, then $S(A, B)$ is in wff^- .
 - (c) If both A and B are in wff^* , then so are $A \wedge B, A \vee B, A \Rightarrow B$, where wff^* is one of wff^+, wff^- or wff^0 .
5. $\neg A$ is also a wff and it is of the same type and has the same free variables as A .
6. $\top$ and $\perp$ are in wff^0 with no free variables.
7. If A is a formula in $wff^*\{x, y_1, \dots, y_m\}$ then $\forall x A$ and $\exists x A$ are wffs in $wff^*\{y_1, \dots, y_m\}$.
8. If $(\varphi q)A(q, q_1, \dots, q_n)$ is a pure past formula of propositional USF as defined in Definition 102, and if $B_i \in wff V_i^*, i = 1, \dots, m$, as defined in the present Definition 103, then $A' = (\varphi q)A(q, B_1, \dots, B_m)$ is a wff in $wff \bigcup_{i=1}^m V_i^*$. If all of the B_i are pure past, then so is A' .

9. A wff A is said to be *essentially propositional* iff there exists a wff $B(q_1, \dots, q_n)$ of propositional USF and wffs $B_1, \dots, B_n$ of classical predicate logic such that $A = B(B_1, \dots, B_n)$.

REMARK 104. Notice that the fixed point operator (φx) is used in propositional USF to define the new connectives, and after it is defined it is exported to the predicate USF. We can define a language HTL which will allow fixed-point operations on predicates as well; we will not discuss this here.

DEFINITION 105. We define the semantic interpretation of propositional USF in the monadic theory of $(\mathbb{N}, <, =, 0)$. An assignment h is a function associating with each atom q_i of USF a subset $h(q_i)$ of $\mathbb{N}$ (sometimes denoted by Q_i). h can be extended to any wff of USF as follows:

$$\begin{aligned} h(A \wedge B) &= h(A) \cap h(B); \\ h(A \vee B) &= h(A) \cup h(B); \\ h(\neg A) &= \mathbb{N} - h(A); \\ h(A \Rightarrow B) &= (\mathbb{N} - h(A)) \cup h(B); \\ h(U(A, B)) &= \{t | \exists s > t (s \in h(A) \text{ and } \forall y (t < y < s \Rightarrow y \in h(B)))\}; \\ h(S(A, B)) &= \{t | \exists s < t (s \in h(A) \text{ and } \forall y (s < y < t \Rightarrow y \in h(B)))\}. \end{aligned}$$

The meaning of U and S just defined is the Until and Since of English, i.e. ‘ B is true until A becomes true’ and ‘ B is true since A was true’, as in Fig. 11 (notice the existential meaning in U and S).

Finally we have the fixed point operator

$$h((\varphi q)A(q, q_i)) = \{n | n \in Q_n\}$$

where the sets $Q_n \subseteq \mathbb{N}$ are defined inductively by

$$\begin{aligned} Q_0 &= h(A) \\ Q_{(n+1)} &= h_n(A(q, q_i)) \end{aligned}$$

where for $n \geq 0$

$$\begin{aligned} h_n(r) &= h(r) \quad \text{for } r \neq q \\ &= Q_n \quad \text{for } r = q. \end{aligned}$$

This is an inductive definition. If n is a natural number, we assume inductively that we know the truth values of the formula $(\varphi q)A(q, q_i)$ at each $m < n$; then we obtain its value at n by first changing the assignment so that q has the same values as $(\varphi q)A$ for all $m < n$, and then taking the new value of $A(q, q_i)$ at n . Since A is pure past, the values of q at $m \geq n$ do not matter. Hence the definition is sound. So $(\varphi q)A$ is defined in terms of its own previous values.

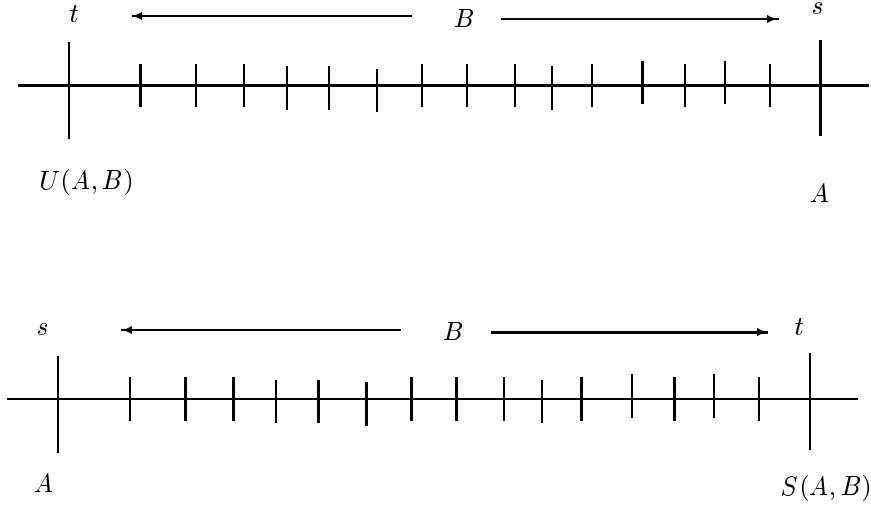


Figure 11.

This gives a fixed point semantics to formulae $(\varphi q)A(q, q_i)$, in the following sense. Suppose we have an assignment h . For any subset S of $\mathbb{N}$, let h_S be the assignment given by $h_S(r) = h(r)$ if $r \neq q$, and S if $r = q$. Then given A as above, we obtain a function $f : \wp\mathbb{N} \Rightarrow \wp\mathbb{N}$, given by $f(S) = h_S(A)$. f depends on h and A .

It is intuitively clear from the above that if $S = h((\varphi q)A)$ then $f(S) = S$, and that S is the unique solution of $f(x) = x$. So $h((\varphi q)A)$ is the unique fixed point of f . This is what we mean when we say that φ has a fixed point semantics. There are some details to be checked, in particular that the value at n of any past formula (even a complicated one involving φ) depends only on the values of its atoms at each $m < n$. For a full proof see [Hodkinson, 1989].

DEFINITION 106 (Semantic definition of predicate USF). Let D be a non-empty set, called the domain, and g be a function assigning the following:

1. for each m -place function symbol f and each $n \in \mathbb{N}$ a function $g(n, f) : D^m \Rightarrow D$;
2. for each variable x and each $n \in \mathbb{N}$, an element $g(n, x) \in D$;
3. for each m -place predicate symbol Q and each $n \in \mathbb{N}$, a function $g(n, Q) : D^m \Rightarrow \{0, 1\}$.

The function g can be extended to a function $g(n, A)$, giving a value in $\{0, 1\}$ for each wff $A(x_1, \dots, x_n)$ of the predicate USF as follows:

1. $g(n, f(t_1, \dots, t_m)) = g(n, f)(g(n, t_1), \dots, g(n, t_m))$;
2. $g(n, Q(t_1, \dots, t_m)) = g(n, Q)(g(n, t_1), \dots, g(n, t_m))$;
3. $g(n, A \wedge B) = 1$ iff $g(n, A) = 1$ and $g(n, B) = 1$;
4. $g(n, A \vee B) = 1$ iff either $g(n, A) = 1$ or $g(n, B) = 1$ or both;
5. $g(n, A \Rightarrow B) = 1$ iff either $g(n, A) = 0$ or $g(n, B) = 1$ or both;
6. $g(n, \neg A) = 1$ iff $g(n, A) = 0$;
7. $g(n, \top) = 1$ and $g(n, \perp) = 0$ for all n ;
8. $g(n, U(A, B)) = 1$ iff for some $m > n$, $g(m, A) = 1$ and for all $n < k < m$, $g(k, B) = 1$;
9. $g(n, S(A, B)) = 1$ iff for some $m < n$, $g(m, A) = 1$ and for all $m < k < n$, $g(k, B) = 1$;
10. $g(n, \forall x A(x)) = 1$ for a variable x iff for all g' such that g' gives the same values as g to all function symbols and all predicate symbols and all variables different from x , we have $g'(n, A(x)) = 1$;
11. $g(n, \exists x A(x)) = 1$ for a variable x iff for some g' such that g' gives the same values as g to all function symbols and all predicate symbols and all variables different from x , we have $g'(n, A(x)) = 1$;
12. let $(\varphi q)A(q, q_1, \dots, q_m)$ be a pure past formula of propositional USF, and let $B_i \in wffV_i^*$ for $i = 1, \dots, m$. We want to define $g(n, A')$, where $A' = (\varphi q)A(q, B_1, \dots, B_m)$. First choose an assignment h such that $h(q_i) = \{n \in \mathbb{N} | g(n, B_i) = 1\}$. Then define $g(n, A') = 1$ iff $n \in h((\varphi q)A(q, q_1, \dots, q_m))$.

REMARK 107. If we let $h_g^*(A)$ be the set $\{n | g(n, A) = 1\}$ we get a function h^* like that of Definition 105.

EXAMPLE 108. Let us evaluate $(\varphi x)A(x)$ for $A(x) = H\neg x$, where $Hx = \neg S(\neg x, \top)$; see Example 109.1. We work out the value of $(\varphi x)A(x)$ at each n , by induction on n . If we know its values for all $m < n$, we assume that the atom x has the same value as $(\varphi x)A(x)$ for $m < n$. We then calculate the value of $A(x)$ at n . So, really, $(\varphi x)A(x)$ is a definition by recursion.

Since $H\neg x$ is a pure past formula, its value at 0 is known and does not depend on x . Thus $A(x)$ is true at 0. Hence $(\varphi x)A(x)$ is true at 0.

Let us compute $A(x)$ at 1. Assume that x is true at 0. Since $A(x)$ is pure past, its value at 1 depends on the value of x at 0, which we know. It

does not depend on the value of x at $n \geq 1$. Thus at 1, $(\varphi x)A(x) = A(x) = H\neg\top = \perp$.

Assume inductively that we know the values of $(\varphi x)A(x)$ at $0, 1, \dots, n$, and suppose that x also has these values at $m \leq n$. We compute $A(x)$ at $n+1$. This depends only on the values of x at points $m \leq n$, which we know. Hence $A(x)$ at $n+1$ can be computed; for our example we get $\perp$. So $(\varphi x)A(x)$ is false at $n+1$. Thus $(\varphi x)H\neg x$ is (semantically) equivalent to $H\perp$, because $H\perp$ is true at 0 and nowhere else.

Another way to get the answer is to use the fixed point semantics directly. Let $f(S) = h(A)$, where $h(x) = S$, as above. Then by definition of f and g ,

$$\begin{aligned} f(S) &= \{n \in \mathbb{N} \mid \neg \exists m < n (m \in S \wedge \forall k (m < k < n \Rightarrow k \in h(\top)))\} \\ &= \{n \in \mathbb{N} \mid \forall m < n (m \notin S)\}. \end{aligned}$$

So $f(S) = S$ iff $S = \{0\}$. Hence the fixed point is $\{0\}$, as before.

Let us evaluate $(\varphi x)B(x)$ where $B(x) = S(S(x, a), \neg a)$. At time 0 the value of $B(x)$ is $\perp$. Let x be $\perp$ at 0. At time 1 the value of $B(x)$ is $S(S(\perp, a), \neg a) = S(\perp, \neg a) = \perp$. Let x be $\perp$ at 1 etc. . . . It is easy to see that $(\varphi x)B(x)$ is independent of a and is equal to $\perp$.

EXAMPLE 109. We give examples of connectives definable in this system.

1. The basic temporal connectives are defined as follows:

Connective	Meaning	Definition
$\bullet q$	q was true ‘yesterday’	$S(q, \perp)$
Xq	q will be true ‘tomorrow’	$U(q, \perp)$
Gq	q ‘will always’ be true	$\neg U(\neg q, \top)$
Fq	q ‘will sometimes’ be true	$U(q, \top)$
Hq	q ‘was always’ true	$\neg S(\neg q, \top)$
Pq	q ‘was sometimes’ true	$S(q, \top)$

Note that at 0, both $\bullet q$ and Pq are false.

2. The first time point (i.e. $n = 0$) can be identified as the point at which $H\perp$ is true.
3. The fixed point operator allows us to define non-first-order definable subsets. For example, $e = (\varphi x)(\bullet \bullet x \vee H\perp)$ is a constant true exactly at the even points $\{0, 2, 4, 6, \dots\}$.
4. $S(A, B)$ can be defined from $\bullet$ using the fixed point operator.:

$$S(A, B) = (\varphi x)(\bullet A \vee \bullet (x \wedge B))'.$$

5. If we have

$$\text{block}(a, b) \triangleq (\varphi x)S(b \wedge S(a \wedge (x \vee H\perp \vee HH\perp), a), b)$$

then $\text{block}(a, b)$ says that we have the sequence of the form

$$(\text{block of } bs) + (\text{block of } as) + \dots$$

recurring in the pure past, beginning yesterday with b and going into the past. In particular $\text{block}(a, b)$ is false at time 0 and time 1 because the smallest recurring block is (b, a) which requires two points in the past.

DEFINITION 110 (Expressive power of USF). Let $\Psi(t, Q_1, \dots, Q_n)$ be a formula in the monadic language of $(\mathbb{N}, <, =, 0, Q_1, \dots, Q_n \subseteq \mathbb{N})$. Let $Q = \{t \mid \Psi(t, Q_i) \text{ is true}\}$. Then Q is said to be monadic first-order definable from Q_i .

EXAMPLE 111. $\text{even} = \{0, 2, 4, \dots\}$ is not monadic first-order definable from any family of finite or cofinite subsets. It is easy to see that every quantificational wff $\Psi(t, Q_i)$, with Q_i finite or cofinite subsets of $\mathbb{N}$, defines another finite or co-infinite subset. But even is definable in USF (Example 109.3 above). even is also definable in monadic second-order logic. In fact, given any formula $A(q_1, \dots, q_n)$ of USF, we can construct a formula $A'(x, Y_1, \dots, Y_n)$ of monadic second-order logic in the language with relations $\subseteq$ and $<$, such that for all h ,

$$h(A) = \{m \mid A'(m, h(q_1), \dots, h(q_n)) \text{ holds in } \mathbb{N}\}.$$

(See [Hodkinson, 1989].) Since this monadic logic is decidable, we get the following theorem.

THEOREM 112. *Propositional USF is decidable. In other words, the set of wffs $\{A \mid h(A) = \mathbb{N} \text{ for all } h\}$ is recursive. See for example [Hodkinson, 1989].*

THEOREM 113. *Many nested applications of the fixed point operator are no stronger than a single one. In fact, any pure past wff of USF is semantically equivalent to a positive Boolean combination (i.e. using $\wedge, \vee$ only) of wffs of the form $(\varphi x)A$, where A is built from atoms using only the Boolean connectives and $\bullet$ (as in Example 109.1,4). See [Hodkinson, 1989].*

THEOREM 114 (Full expressiveness of S and U [Kamp, 1968a]). *Let $Q_1, \dots, Q_n \subseteq \mathbb{N}$ be n set variables and let $\Psi(t, Q_1, \dots, Q_n)$ be a first-order monadic formula built up from $Q_1, \dots, Q_n$ using $<, =$ and the quantifiers over elements of $\mathbb{N}$ and Boolean connectives. Then there exists a wff of USF, $A_\Psi(q_1, \dots, q_n)$, built up using S and U only (without the use of the fixed point operator φ) such that for all h and all Q_i the following holds:*

$$\text{If } h(q_i) = Q_i \text{ then } h(A_\Psi) = \{t \mid \Psi(t, Q_i) \text{ holds in } \mathbb{N}\}.$$

Proof. H. Kamp proved this theorem directly by constructing A_Ψ . Another proof was given in [Gabbay *et al.*, 1994]. The significance of this theorem is that S and U alone are exactly as expressive (as a specification language) as first-order quantification $\forall, \exists$ over temporal points. The use of φ takes USF beyond first-order quantification. ■

THEOREM 115 (Well known). *Predicate USF without fixed point applications is not arithmetical ([Kamp, 1968a]).*

8.2 USF as a specification language

The logic USF can be used as a specification language as follows. Let $A(a_1, \dots, a_m, b_1, \dots, b_k)$ be a wff of USF. Let h be an assignment to the atoms $\{a_i, b_j\}$. We say that h satisfies the specification A iff $h(A) = \mathbb{N}$.

In practice, the atoms b_j are controlled by the environment, and the atoms a_i are controlled by the program. Thus the truth values of b_j are determined as *events* and the truth values of a_i are determined by the program *execution*. As time moves forward and the program interacts with the environment, we get a function h , which may or may not satisfy the specification.

Let $\text{event}(q, n)$ mean that the value of q at time n is truth, as determined by the environment and let $\text{exec}(q, n)$ mean that q is executed at time n and therefore the truth value of q at time n is *true*. Thus out of event and exec we can get a full assignment $h = \text{event} + \text{exec}$ by letting:

$$\begin{aligned} h(q) &= \{n \mid \text{event}(q, n) \text{ holds}\} \text{ for } q \text{ controlled by the environment;} \\ h(q) &= \{n \mid \text{exec}(q, n) \text{ holds}\} \text{ for } q \text{ controlled by the program.} \end{aligned}$$

Of course our aim is to execute in such a way that the h obtained satisfies the specification.

We now explain how to execute any wff of our temporal language. Recall that the truth values of the atoms a_i come from the program *via* $\text{exec}(a_i, m)$ and the truth values for the atoms b_j come from the environment via the function $\text{event}(b_j, m)$. We define a predicate $\text{exec}^*(A, m)$ for any wff A , which actually defines the value of A at time m .

For atoms of the form a_i , $\text{exec}^*(a_i, m)$ will be $\text{exec}(a_i, m)$ and for atoms of the form b_j (i.e. controlled by the environment) $\text{exec}^*(b_j, m)$ will be $\text{event}(b_j, m)$. We exec^* an atom controlled by the environment by ‘agreeing’ with the environment. For the case of A a pure past formula, $\text{exec}^*(A, m)$ is determined by past truth values. Thus exec^* for pure past sentences is really a *hold* predicate, giving truth values determined already. For pure future sentences B , $\text{exec}^*(B, m)$ will have an operational meaning. For example, we will have

$$\text{exec}^*(G \text{ print}, m) = \text{exec}^*(\text{print}, m) \wedge \text{exec}^*(G \text{ print}, m + 1).$$

We can assume that the wffs to be executed are pure formulae (pure past, pure present or pure future) such that all negations are pushed next to atoms. This can be done because of the following semantic equivalences.

1. $\neg U(a, b) = G\neg a \vee U(\neg b \wedge \neg a, \neg a);$
2. $\neg S(a, b) = H\neg a \vee S(\neg b \wedge \neg a, \neg a);$
3. $\neg Ga = F\neg a;$
4. $\neg Ha = P\neg a;$
5. $\neg Fa = G\neg a;$
6. $\neg Pa = H\neg a;$
7. $\neg[(\varphi x)A(x)] = (\varphi x)\neg A(\neg x).$

See [Gabbay *et al.*, 1994] for 1 to 6. For 7, let h be an assignment and suppose that $h(\neg[(\varphi x)A(x)]) = S$. Because φ has fixed point semantics, to prove 7 it is enough to show that if h' is the assignment that agrees with h on all atoms except x , and $h'(x) = S$, then $h'(\neg(A\neg x)) = S$. Clearly, $h(\neg[(\varphi x)A(x)]) = \mathbb{N} \setminus S$. We may assume that $h(x) = \mathbb{N} \setminus S$. Then $h(A(x)) = \mathbb{N} \setminus S$. So $h(\neg A(x)) = S$ and $h'(\neg(A\neg x)) = S$ as required; 7 is also easy to see using the recursive approach.

DEFINITION 116. We assume that $\text{exec}^*(A, m)$ is defined for the system for any m and any A which is atomic or the negation of an atom. For atomic b which is controlled by the environment, we assume $\text{event}(b, m)$ is defined and $\text{exec}^*(b, m) = \text{event}(b, m)$. For $\text{exec}^*(a, m)$, for a controlled by the program, execution may be done by another program. It may be a graphical or a mathematical program. It certainly makes a difference what m is relative to now. If we want to $\text{exec}^*(a, m)$ for m in the past of now, then a has already been executed (or not) and so $\text{exec}^*(a, m)$ is a hold predicate. It agrees with what has been done. Otherwise (if $m \geq$ now) we do execute*.

1. $\text{exec}^*(\top, m) = \top.$
2. $\text{exec}^*(\perp, m) = \perp.$
3. $\text{exec}^*(A \wedge B, m) = \text{exec}^*(A, m) \wedge \text{exec}^*(B, m).$
4. $\text{exec}^*(A \vee B, m) = \text{exec}^*(A, m) \vee \text{exec}^*(B, m).$
5. $\text{exec}^*(S(A, B), 0) = \perp.$
6. $\text{exec}^*(S(A, B), m+1) = \text{exec}^*(A, m) \vee [\text{exec}^*(B, m) \wedge \text{exec}^*(S(A, B), m)].$

7. $\text{exec}^*(U(A, B), m) = \text{exec}^*(A, m+1) \vee [\text{exec}^*(B, m+1) \wedge \text{exec}^*(U(A, B), m+1)]$.
8. $\text{exec}^*((\varphi x)A(x), 0) = A_0$, where A_0 is obtained from A by substituting $\top$ for any wff of the form HB and $\perp$ for any wff of the form PB or $S(B_1, B_2)$.
9. $\text{exec}^*((\varphi x)A(x), m+1) = \text{exec}^*(A(C), m+1)$, where C is a new atom defined for $n \leq m$ by $\text{exec}^*(C, n) = \text{exec}^*((\varphi x)A(x), n)$. In other words $\text{exec}^*((\varphi x)A(x), m+1) = \text{exec}^*(A((\varphi x)A(x)), m+1)$ and since in the execution of A at time $m+1$ we go down to executing A at time $n \leq m$, we will have to execute $(\varphi x)A(x)$ at $n \leq m$, which we assume by induction that we already know.
10. In the predicate case we can let

$$\begin{aligned} \text{exec}^*(\forall y A(y)) &= \forall y \text{exec}^*(A(y)) \\ \text{exec}^*(\exists y A(y)) &= \exists y \text{exec}^*(A(y)). \end{aligned}$$

We are now in a position to discuss how the execution of a specification is going to be carried out in practice. Start with a specification $\mathcal{S}$. For simplicity we assume that $\mathcal{S}$ is written in essentially propositional USF which means that $\mathcal{S}$ contains S , U and φ operators applied to pure past formulae, and is built up from atomic units which are wffs of classical logic. If we regard any fixed point wff $(\varphi x)D(x)$ as atomic, we can apply the separation theorem and rewrite $\mathcal{S}$ into an executable form $\mathcal{E}$, which is a conjunction of formulae such that

$$\Omega \equiv \bigwedge_k \left[\bigwedge_i C_{i,k} \Rightarrow \bigvee_j B_{j,k} \right]$$

where $C_{i,k}$ are pure past formulae (containing S only) and $B_{j,k}$ are either atomic or pure future formulae (containing U). However, since we regarded any (φx) formula as an atom, the $B_{j,k}$ can contain $(\varphi x)D(x)$ formulae in them. Thus $B_{j,k}$ can be for example $U(a, (\varphi x)[\bullet \neg x])$. We will assume that any such $(\varphi x)D(x)$ contains only atoms controlled by the environment; this is a restriction on $\mathcal{E}$. Again, this is because we have no separation theorem as yet for full propositional USF, but only for the fragment US of formulae not involving φ . We conjecture that—possibly in a strengthened version of USF that allows more fixed point formulae—any formula can be separated. This again remains to be done.

However, even without such a result we can still make progress. Although $(\varphi x)[\bullet \neg x]$ is a pure past formula within U , it is still an executable formula that only refers to environment atoms, and so we do not mind having it there. If program atoms were involved, we might have a formula equivalent

to $X \bullet \text{print}$ (say), so that we would have to execute $\bullet \text{print}$ tomorrow. This is not impossible: when tomorrow arrives we check whether we did in fact print yesterday, and return $\top$ or $\perp$ accordingly. But it is not a very intelligent way of executing the specification, since clearly we should have just printed in the first instance. This illustrates why we need to separate $\mathcal{S}$.

Recall the equation for executing $U(A, B)$:

$$\text{exec}^*(U(A, B)) \equiv X \text{exec}^*(A) \vee (X(\text{exec}^*(B) \wedge \text{exec}^*(U(A, B))).$$

If either A or B is of the form $(\varphi x)D(x)$, we know how to compute $\text{exec}^*((\varphi x) D(x))$ by referring to past values. Thus $(\varphi x)D(x)$ can be regarded as atomic because we know how to execute it, in the same way as we know how to execute **write**.

Imagine now that we are at time n . We want to make sure the specification $\mathcal{E}$ remains true. To keep $\mathcal{E}$ true we must keep true each conjunct of $\mathcal{E}$. To keep true a conjunct of the form $C \Rightarrow B$ where C is past and B is future, we check whether C is true in the past. if it is true, then we have to make sure that B is true in the future. *Since the future has not happened yet, we can read B imperatively, and try to force the future to be true.* Thus the specification $C \Rightarrow B$ is read by us as

$$\text{hold}(C) \Rightarrow \text{exec}^*(B).$$

Some future formulae cannot be executed immediately. We already saw that to execute $U(A, B)$ now we either execute A tomorrow or execute B tomorrow together with $U(A, B)$. Thus we have to pass a list of formulae to execute from today to tomorrow. Therefore at time $n + 1$, we have a list of formulae to execute which we inherit from time n , in addition to the list of formulae to execute at time $n + 1$. We can thus summarize the situation at time $n + 1$ as follows:

1. Let $G_1, \dots, G_m$ be a list of wffs we have to execute at time $n + 1$. Each G_i is a disjunction of formulae of the form atomic or negation of atomic or FA or GA or $U(A, B)$.
2. In addition to the above, we are required to satisfy the specification $\mathcal{E}$, namely

$$\bigwedge_k \left[\bigwedge_i C_{i,k} \Rightarrow \bigvee_j B_{j,k} \right]$$

for each k such that $\bigwedge_i C_{i,k}$ holds (in the past). We must execute the future (and present) formula $B_k = \bigvee_j B_{j,k}$ which is again a disjunction of the same form as in 1 above.

We know how to execute a formula; for example,

$$\text{exec}^*(FA) = X\text{exec}^*A \vee X \text{exec}^*(FA).$$

FA means ‘ A will be true’. To execute FA we can either make A true tomorrow or make FA true tomorrow. What we should be careful not to do is not to keep on executing FA day after day because this way A will never become true. Clearly then we should try to execute A tomorrow and if we cannot, only then do we execute FA by doing $X \text{exec}^*(FA)$. We can thus read the disjunction $\text{exec}^*(A \vee B)$ as *first* try to exec^*A and *then only if we fail* exec^*B . This priority (left to right) is not a logical part of ‘ $\vee$ ’ but a procedural addition required for the correctness of the model. We can thus assume that the formulae given to execute at time n are written as disjunctions with the left disjuncts having priority in execution. Atomic sentences or their negations always have priority in execution (though this is not always the best practical policy).

Let $D = \bigvee_j D_j$ be any wff which has to be executed at time $n+1$, either because it is inherited from time n or because it has to be executed owing to the requirements of the specification at time $n+1$. To execute D , either we execute an atom and discharge our duty to execute, or we pass possibly several disjunctions to time $n+2$ to execute then (at $n+2$), and the passing of the disjunctions will discharge our obligation to execute D at time $n+1$. Formally we have

$$\text{exec}^*(D) = \bigvee_j \text{exec}^*(D_j).$$

Recall that we try to execute left to right. The atoms and their negations are supposed to be on the left. If we can execute any of them we are finished with D . If an atom is an environment atom, we check whether the environment gives it the right value. If the atom is under the program’s control, we can execute it. However, the negation of the atom may appear in another formula D' to be executed and there may be a clash. See Examples 117 and 118 below. At any rate, should we choose to execute an atom or negation of an atom and succeed in doing so, then we are finished. Otherwise we can execute another disjunct of D of the form $D_j = U(A_j, B_j)$ or of the form GA_j or FA_j . We can pass the commitment to execute to the time $n+2$. Thus we get

$$\text{exec}^*(D) = \bigvee \text{exec}^*(\text{atoms of } D) \vee \text{exec}^*(\text{future formulae of } D).$$

Thus if we cannot execute the atoms at time $n+1$, we pass to time $n+2$ a conjunction of disjunctions to be executed, ensuring that atoms and subformulae should be executed before formulae. We can write the disjunctions to reflect these priorities. Notice further that although, on first impression, the formulae to be executed seem to multiply, they actually do not.

At time $n = 0$ all there is to execute are heads of conditions in the specification. If we cannot execute a formula at time 0 then we pass execution to time 1. This means that at time 1 we inherit the execution of

$A \vee (B \wedge U(A, B))$, where $U(A, B)$ is a disjunct in a head of the specification. This same $U(A, B)$ may be passed on to time 2, or some subformula of A or B may be passed. The number of such subformulae is limited and we will end up with a limited stock of formulae to be passed on. In practice this can be optimized. We have thus explained how to execute whatever is to be executed at time n . When we perform the execution sequence at times $n, n+1, n+2, \dots$, we see that there are now two possibilities:

- We cannot go on because we cannot execute all the demands at the same time. In this case we stop. The specification cannot be satisfied either because it is a contradiction or because of a wrong execution choice (e.g. we should not have printed at time 1, as the specification does not allow anything to be done after printing).
- Another possibility is that we see after a while that the same formulae are passed for execution from time n to time $n+1$ to $n+2$ etc. This is a loop. Since we have given priority in execution to atoms and to the A in $U(A, B)$, such a loop means that it is not possible to make a change in execution, and therefore either the specification cannot be satisfied because of a contradiction or wrong choice of execution, or the execution is already satisfied by this loop.

EXAMPLE 117. All atoms are controlled by the program. Let the specification be

$$Ga \wedge F\neg a.$$

Now the rules to execute the subformulae of this specification are

$$\begin{aligned} \text{exec}^*(Ga) &\equiv \text{exec}^*(a) \wedge \text{exec}^*(Ga) \\ \text{exec}^*(F\neg a) &\equiv \text{exec}^*(\neg a) \vee \text{exec}^*(F\neg a). \end{aligned}$$

To execute Ga we must execute a . Thus we are forced to discharge our execution duty of $F\neg a$ by passing $F\neg a$ to time $n+1$. Thus time $n+1$ will inherit from time n the need to execute $Ga \wedge F\neg a$. This is a loop. The specification is unsatisfiable.

EXAMPLE 118. The specification is

$$b \vee Ga$$

$$Pb \Rightarrow F\neg a \wedge Ga.$$

According to our priorities we execute b first at time 0. Thus we will have to execute $F\neg a \wedge Ga$ at time 1, which is impossible. Here we made the wrong execution choice. If we keep on executing $\neg b \wedge Ga$ we will behave as specified.

In practice, since we may have several choices in execution we may want to simulate the future a little to see if we are making the correct choice.

Having defined **exec***, we need to add the concept of updating. Indeed, the viability of our notion of the declarative past and imperative future depends on adding information to our database. In this section we shall assume that every event that occurs in the environment, and every action **exec**-ed by our system, are recorded in the database. This is of course unnecessary, and in a future paper we shall present a more realistic method of updating.

8.3 The logic *USF2*

The fixed point operator that we have introduced in propositional USF has to do with the solution of the equation

$$x \leftrightarrow B(x, q_1, \dots, q_m)$$

where B is a pure past formula. Such a solution always exists and is unique. The above equation defines a connective $A(q_1, \dots, q_m)$ such that

$$A(q_1, \dots, q_m) \leftrightarrow B(A(q_1, \dots, q_m), q_1, \dots, q_m).$$

Thus, for example, $S(p, q)$ is the solution of the equation

$$x \leftrightarrow \bullet p \vee \bullet (q \wedge x)$$

as we have $S(p, q) \leftrightarrow \bullet p \vee \bullet (q \wedge S(p, q))$. Notice that the connective to be defined ($x = S(p, q)$) appears as a unit in both sides of the equation.

To prove existence of a solution we proceed by induction. Suppose we know what x is at time $\{0, \dots, n\}$. To find what x is supposed to be at time $n + 1$, we use the equation $x \leftrightarrow B(x, q_i)$. Since B is pure past, to compute B at time $n + 1$ we need to know $\{x, q_i\}$ at times $\leq n$, which we do know. This is the reason why we get a unique solution.

Let us now look at the following equation for a connective $Z(p, q)$. We want Z to satisfy the equation

$$Z(p, q) \leftrightarrow \bullet p \vee \bullet (q \wedge Z(\bullet p, q)).$$

Here we did not take $Z(p, q)$ as a unit in the equation, but substituted a value $\bullet p$ in the right-hand side, namely $Z(\bullet p, q)$. $\bullet p$ is a pure past formula. We can still get a unique solution because $Z(p, q)$ at time $n + 1$ still depends on the values of $Z(p, q)$ at earlier times, and certainly we can compute the values of $Z(\bullet p, q)$ at earlier times.

The general form of the new fixed point equation is as follows:

DEFINITION 119 (Second-order fixed points). Let $Z(q_1, \dots, q_m)$ be a candidate for a new connective to be defined. Let $B(x, q_1, \dots, q_m)$ be a pure

past formula and let $D_i(q_1, \dots, q_m)$ for $i = 1 \dots m$ be arbitrary formulae. Then we can define Z as the solution of the following equation:

$$Z(q_1, \dots, q_m) \leftrightarrow B(Z[D_1(q_1, \dots, q_m), \dots, D_m(q_1, \dots, q_m)], q_1, \dots, q_m).$$

We call this definition of Z *second order*, because we can regard the equation as

$$Z \equiv \text{Application}(Z, D_i, q_j).$$

We define USF2 to be the logic obtained from USF by allowing nested applications of second-order fixed point equations. USF2 is more expressive than USF (Example 120).

Predicate USF2 is defined in a similar way to predicate USF.

EXAMPLE 120. Let us see what we get for the connective $Z_1(p, q)$ defined by the equation

$$Z_1(p, q) \leftrightarrow \bullet p \vee \bullet (q \wedge Z_1(\bullet p, q)).$$

The connective $Z_1(p, q)$ says what is shown in Fig.12:

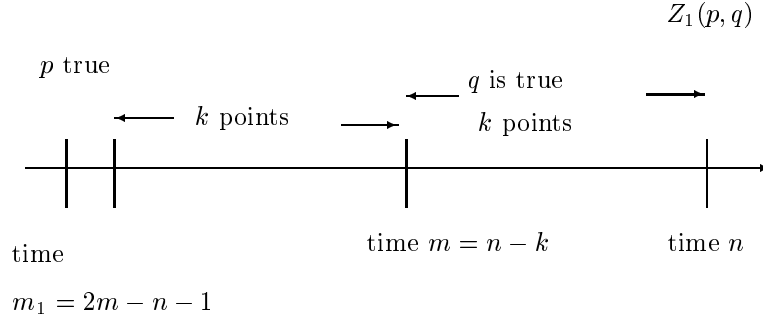


Figure 12.

$Z_1(p, q)$ is true at n iff for some $m \leq n$, q is true at all points j with $m \leq j < n$, and p is true at the point $m_1 = m - (n - m + 1) = 2m - n - 1$. If we let $k = n - m$, then we are saying that q is true k times into the past and before that p is true at a point which is $k + 1$ times further into the past. This is not expressible with any pure past formula of USF; see [Hodkinson, 1989].

Let us see whether this connective satisfies the fixed point equation

$$Z_1(p, q) \leftrightarrow \bullet p \vee \bullet (q \wedge Z_1(\bullet p, q)).$$

If $\bullet p$ is true then $k = 0$ and the definition of $Z_1(p, q)$ is correct. If $\bullet (q \wedge Z_1(\bullet p, q))$ is true, then we have for some k the situation in Fig. 13:

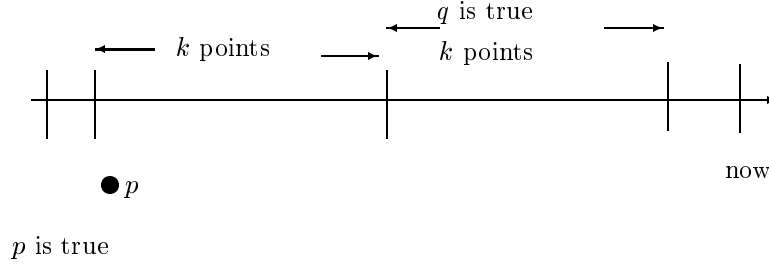


Figure 13.

The definition of $Z_1(p, q)$ is satisfied for $k + 1$.

EXAMPLE 121 (Coding of dates). We can encode dates in the logic as follows:

1. The proposition $\neg \bullet \top$ is true exactly at time 0, since it says that there is no yesterday. Thus if we let

$$\mathbf{n} = \perp \text{ if } n \leq 0$$

$$0 = \neg \bullet \top$$

$$\mathbf{n} = \bullet(\mathbf{n} - \mathbf{1}).$$

then we have that $\mathbf{n}$ is true exactly at time n . This is a way of naming time n . In predicate temporal logic we can use elements to name time. Let $\text{date}(x)$ be a predicate such that the following hold at all times n :

$$\exists x \text{ date}(x)$$

$$\forall x(\text{date}(x) \Rightarrow G\neg \text{date}(x) \wedge H\neg \text{date}(x))$$

$$\forall x(\text{date}(x) \vee P \text{date}(x) \vee F \text{date}(x)).$$

These axioms simply say that each time n is identified by some element x in the domain that uniquely makes $\text{date}(x)$ true, and every domain element corresponds to a time.

2. We can use this device to count in the model. Suppose we want to define a connective that counts how many times A was true in the past. We can represent the number m by the date formula $\mathbf{m}$, and define $\text{count}(A, \mathbf{m})$ to be true at time n iff the number of times before n in which A was true is exactly $\mathbf{m}$. Thus in Fig. 14, $\text{count}(A, \bullet \top \wedge \neg \bullet \bullet \top)$ is false at time 3, true at time 2, true at time 1 and false at time 0.

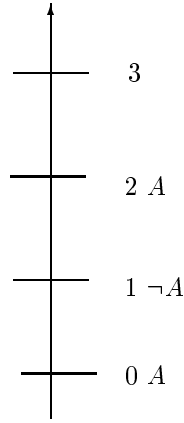


Figure 14.

The connective **count** can be defined by recursion as follows:

$$\begin{aligned} \text{count}(p, \mathbf{n}) \quad \leftrightarrow \quad & \bullet (\neg p \wedge \text{count}(p, \mathbf{n})) \\ & \vee \bullet (p \wedge \text{count}(p, X\mathbf{n})) \\ & \vee (\neg \bullet \top \wedge \mathbf{n}). \end{aligned}$$

Note that $X\mathbf{n}$ is equivalent to $\mathbf{n} - \mathbf{1}$. We have cheated in this example. For the formula $B(x, q_1, q_2)$ in the definition of second-order fixed points is here

$$\bullet (\neg q_1 \wedge x) \vee \bullet (q_1 \wedge x) \vee (\neg \bullet \top \wedge q_2).$$

This is not pure past, as q_2 occurs in the present tense. To deal with this we could define the notion of a formula $B(x, q_1, \dots, q_m)$ being pure past in x . See [Hodkinson, 1989]. We could then amend the definition to allow any B that is pure past in x . This would cover the B here, as all x s in B occur under a $\bullet$. So the value of the connective at n still depends only on its values at $m \leq n$, which is all we need for there to be a fixed point solution. We do not do this formally here, as we can express count in standard USF2; see the next example.

EXAMPLE 122. We can now define the connective **more**(A, B) reading ‘ A was true more times than B ’.

$$\begin{aligned} \text{more}(A, B) \quad \leftrightarrow \quad & \bullet (A \wedge \text{more}(A, B)) \\ & \vee \bullet (\neg A \wedge \neg B \wedge \text{more}(A, B)) \\ & \vee \bullet (\neg A \wedge \text{more}((A \wedge PA), B)). \end{aligned}$$

(If $k > 0$, then at any n , $A \wedge PA$ has been true k times iff A has been true $k + 1$ times.)

Note that for any $k > 0$, the formula $E_k = \neg \bullet^k \top$ is true exactly k times, at $0, 1, \dots, k - 1$. If we define

$$\text{count}^*(p, k) = \text{more}(E_{k+1}, p) \wedge \neg \text{more}(E_k, p),$$

then at any n , p has been true k times iff $\text{count}^*(p, k)$ holds. So we can do the previous example in standard USF2.

THEOREM 123 (For propositional USF2). *Nested applications of the second-order fixed point operator are equivalent to one application. Any wff A of USF2 is equivalent to a wff B of USF2 built up using no nested applications of the second-order fixed point operator.*

8.4 Payroll example in detail

This section will consider in detail the execution procedures for the payroll example in Section 8.

First let us describe, in the temporal logic USF2, the specification required by Mrs Smith. We translate from the English in a natural way. This is important because we want our logical specification to be readable and have the same structure as in English.

Recall that the intended interpretation of the predicates to be used is

$A(x)$	x is asked to babysit
$B(x)$	x does a babysitting job
$M(x)$	x works after midnight
$P(x, y)$	x is paid y pounds.

‘Babysitters are not allowed to take jobs three nights in a row, or two nights in a row if the first night involves overtime’ is translated as

$$(a) \quad \forall x \neg [B(x) \wedge \bullet B(x) \wedge \bullet \bullet B(x)]$$

$$(b) \quad \forall x \neg [B(x) \wedge \bullet (B(x) \wedge M(x))]$$

$$(c) \quad \forall x [M(x) \Rightarrow B(x)].$$

Note that these wffs are not essentially propositional.

‘Priority in calling is given to those who were not called before as many times as others’ is translated as

$$(d) \quad \neg \exists x \exists y [\text{more}(A(x), A(y)) \wedge A(x) \wedge \neg A(y) \wedge \neg \bullet M(y) \wedge \neg \bullet (B(y) \wedge \bullet B(y))].$$

‘Payment should be made the next day after the job was done, with £15 for a job involving overtime, and £10 for a job not involving overtime’ is translated as

$$(e) \forall x[M(x) \Rightarrow XP(x, 15)]$$

$$(f) \forall x[B(x) \wedge \neg M(x) \Rightarrow XP(x, 10)]$$

$$(g) \forall x[\neg B(x) \Rightarrow X\neg\exists yP(x, y)].$$

Besides the above we also have

$$(h) \forall x[B(x) \Rightarrow A(x)].$$

Babysitters work only when they are called.

We have to rewrite the above into an executable form, namely

Past $\Rightarrow$ Present $\vee$ Future.

We transform the specification to the following:

$$(a') \forall x[\bullet B(x) \wedge \bullet \bullet B(x) \Rightarrow \neg B(x)]$$

$$(b') \forall x[\bullet (B(x) \wedge M(x)) \Rightarrow \neg B(x)]$$

$$(c') \forall x[\neg M(x) \vee B(x)].$$

$$(d') \forall x \forall y[\text{more}(A(x), A(y)) \wedge \neg \bullet M(y) \wedge \neg \bullet (B(y) \wedge \bullet B(y)) \Rightarrow \neg A(x) \vee \neg A(y)]$$

$$(e') \forall x[\neg M(x) \vee XP(x, 15)]$$

$$(f') \forall x[\neg B(x) \vee M(x) \vee XP(x, 10)]$$

$$(g') \forall x[B(x) \vee X\forall y\neg P(x, y)]$$

$$(h') \forall x[\neg B(x) \vee A(x)].$$

Note that **(e')**, **(f')** and **(h')** can be rewritten in the following form using the $\bullet$ operator.

$$(e'') \forall x[\bullet M(x) \Rightarrow P(x, 15)]$$

$$(f'') \forall x[\bullet (B(x) \wedge \neg M(x)) \Rightarrow P(x, 10)]$$

$$(g'') \forall x[\neg \bullet B(x) \Rightarrow \forall y\neg P(x, y)].$$

Our executable sentences become

$$(a^*) \text{hold}(\bullet B(x) \wedge \bullet \bullet B(x)) \Rightarrow \text{exec}(\neg B(x))$$

$$(b^*) \text{hold}(\bullet (B(x) \wedge M(x))) \Rightarrow \text{exec}(\neg B(x))$$

$$(c^*) \text{exec}(\neg M(x) \vee B(x))$$

$$(d^*) \text{hold}(\text{more}(A(x), A(y)) \wedge \neg \bullet M(y) \wedge \neg \bullet (B(y) \wedge \bullet B(y))) \Rightarrow \text{exec}(\neg A(x) \vee \neg A(y))$$

(e*) $\text{exec}(\neg M(x) \vee XP(x, 15))$

(f*) $\text{exec}(\neg B(x) \vee M(x) \vee XP(x, 10))$

(g*) $\text{exec}(B(x) \vee X\forall y\neg P(x, y))$

(h*) $\text{exec}(\neg B(x) \vee A(x))$.

If we use (e''), (f''), (g'') the executable form will be

(e**) $\text{hold}(\bullet M(x)) \Rightarrow \text{exec}(P(x, 15))$

(f**) $\text{hold}(\bullet (B(x) \wedge \neg M(x))) \Rightarrow \text{exec}(P(x, 10))$

(g**) $\text{hold}(\neg \bullet B(x)) \Rightarrow \text{exec}(\forall y\neg P(x, y))$.

In practice there is no difference whether we use (e**) or (e*). We execute XP by sending P to tomorrow for execution. If the specification is (e**), we send nothing to tomorrow but we will find out tomorrow that we have to execute P .

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COMBINATIONS OF TENSE AND MODALITY

1 INTERACTIONS WITH TIME

Physics should have helped us to realise that a temporal theory of a phenomenon X is, in general, more than a simple combination of two components: the statics of X and the ordered set of temporal instants. The case in which all functions from times to world-states are allowed is uninteresting; there are too many such functions, and the theory has not begun until we have begun to restrict them. And often the principles that emerge from the interaction of time with the phenomena seem new and surprising. The most dramatic example of this, perhaps, is the interaction of space with time in relativistic space-time.

The general moral, then, is that we shouldn't expect the theory of time + X to be obtained by mechanically combining the theory of time and the theory of X .¹

Probability is a case that is closer to our topic. Much ink has been spilled over the evolution of probabilities: take, for instance, the mathematical theory of Markov processes (Howard [1971a; 1971b] make a good text), or the more philosophical question of rational belief change (see, for example, Chapter 11 of Jeffrey [1990] and Harper [1975].) Again, there is more to these combinations than can be obtained by separate reflection on probability measure and the time axis.

probability shares many features with modalities and, despite the fact that (classical) probabilities are numbers, perhaps in some sense probability *is* a modality. It is certainly the classic case of the use of possible worlds in interpreting a calculus. (Sample points in a state space are merely possible worlds under another name.) But the literature on probability is enormous, and almost none of it is presented from the logician's perspective. So, aside from the references I have given, I will exclude it from this survey. However, it seems that the techniques we will be using can also help to illuminate problems having to do with probability; this is illustrated by papers such as D. Lewis [1981] and Van Fraassen [1971]. For lack of space, these are not discussed in the present essay.

¹For a treatment that follows this procedure, see [Woolhouse, 1973]; [Werner, 1974] may also fit into this category, but I have not been able to obtain a copy of it. The tense logic of Woolhouse's paper is fairly crude: e.g. moments of time appear both in models and in the object language. The paper seems mainly to be of historical interest.

2 INTRODUCTION TO HISTORICAL NECESSITY

Modern modal logic began with necessity (or with things definable with respect to necessity), and the earliest literature, like C. I. Lewis [1918], confuses this with validity. Even in later work that is formally scrupulous about distinguishing these things, it is sometimes difficult to tell what concepts are really metalinguistic. Carnap, for instance [1956, p. 10], begins his account of necessity by directing our attention to *l*-truth; a sentence of a semantical system (or language) is *L*-true when its truth follows from the semantical rules of the language, without auxiliary assumptions. This, of course, is a metalinguistic notion. But later, when he introduces necessity into the object language [Carnap, 1956, p. 174], he stipulates that $\Box\varphi$ is true if and only if φ is *L*-true.

Carnap thinks of the languages with which he is working as fully determinate; in particular, their semantical rules are fixed. This has the consequence that whatever is *L*-true in a language is eternally *L*-true in that language. (See [Schlipp, 1963, p. 921], for one passage in which Carnap is explicit on the point: he says ‘analytic sentences cannot change their truth-value’.) Combining this consequence with Carnap’s explication of necessity, we see that²

$$(1) \quad \Box\varphi \rightarrow HG\Box\varphi$$

will be valid in languages containing both necessity and tense operators: necessary truths will be eternally true. The combination of necessity with tense would then be trivialised.

But there are difficulties with Carnap’s picture of necessity; indeed, it seems to be drastically misconceived.³ For one thing, many things appear to be necessary, even though the sentences that express them can’t be derived from semantical rules. In Kripke [1982], for instance, published 26 years after *Meaning and Necessity*, Saul Kripke argues that it is necessary that Hesperus is Phosphorous, though ‘Hesperus’ and ‘Phosphorous’ are by no means synonymous. Also at work in Kripke’s conception of necessity, and that of many other contemporaries, is the distinction between φ expressing a necessary truth, and φ necessarily expressing a truth. In a well-known defence of the analytic-synthetic distinction, Grice and Strawson [1956] write as follows:

²I use the tense logical notation of the first Chapter in this volume.

³For an early appreciation of the philosophical importance of making necessity time-dependent (the point I myself am leading up to), see [Lehrer and Taylor, 1965]. The puzzles they raise in this paper are genuine and well presented. But the solution they suggest is very implausible, and the considerations that motivate it seem to confuse semantic and pragmatic phenomena. This is a good example of a case in which philosophical reflections could have been aided by an appeal to the technical apparatus of model theory (in this case, to the model theory of tense logic).

Any form of words at one time held to express something true may, no doubt, at another time come to be held to express something false. but it is not only philosophers who would distinguish between the case where this happens as the result of a change of opinion solely as to matters of fact, and the case where this happens at least partly as a result of shift in the sense of the words (p. 157).

This distinction, at least in theory, makes it possible that a sentence φ should necessarily (perhaps, because of semantical rules) express a truth, even though the truth that it expresses is contingent. This idea is developed most clearly in [Kaplan, 1978].

On this view of necessity, it attaches not primarily to sentences, but to propositions. A sentence will express a proposition, which may or may not be necessary. This can be explicated using possible worlds: propositions take on truth values in these worlds, and a proposition is necessary if and only if it is true in all possible worlds.⁴

This conception can be made temporal without trivialising the results. Probably the simplest way of managing this is to begin with nonempty sets T of times and W of worlds;⁵ T is linearly ordered by a relation $<$. I will call this the $T \times W$ approach.

Recall that a tensed formula, say $F\varphi$, is true at $\langle w, t \rangle$, where $w \in W$ and $t \in T$, if and only if φ is true at $\langle w, t' \rangle$, for some t' such that $t < t'$.⁶ We now want to ask under what conditions $\Box\varphi$ is true at $\langle w, t \rangle$. (In putting it this way we are suppressing propositions; this is legitimate, as long as we treat propositional attitudes as unanalysed, and assume that sentences express the same proposition everywhere.)

If we appeal to intuitions about languages like English, it seems that we should treat formulas like $\Box\varphi$ as nontrivially tensed. This is shown most clearly by sentences involving the adjective 'possible', such as 'In 1932 it was possible for Great Britain to avoid war with Germany; but in 1937 it was impossible'. This suggests that when $\Box\varphi$ is evaluated at $\langle w, t \rangle$ we

⁴To simplify matters, I confine the discussion to the absolute necessity of S5. But perhaps I should mention in passing that in explicating the relative breeds of necessity, such as that of S4, it is easy to confuse modal relations with temporal relations, relative necessity with evanescent necessity. And, of course, tense logic was inspired in part by work on relative necessity. But the two notions are separate; an S5 breed of necessity, for instance, can be evanescent. And when tense and modality are combined, it is very important to attend to the distinction.

⁵I dislike this way of arranging things for philosophical reasons. it doesn't strike me as a logical truth that all worlds have the same temporal orderings: some may have an earliest time, for instance, and others not. Also, the notion of different worlds sharing the same time is philosophically problematic; it is hard to reconcile with a plausible theory of time, when the possible worlds differ widely. Finally, I like to think of possible worlds as overlapping, so that at the same moment may have alternative futures. This requires a more complicated representation. However, the $T \times W$ arrangement will do for now.

⁶See the discussion of the interpretation of tense in Chapter 1 of this volume.

are considering what is *then necessary*; what is true in all worlds at that particular time, t .

The rule then is that $\Box\varphi$ is true at $\langle w, t \rangle$ if and only if φ is true at $\langle w', t \rangle$ for all $w' \in W$. If we like, we can make this relational. Let $\{\approx_t: t \in T\}$ be a family of equivalence relations on W , and let $\Box\varphi$ be true at $\langle t, w \rangle$ if and only if φ is true at $\langle t, w' \rangle$ for all $w' \in W$ such that $w \approx_t w'$.

The resulting theory generates some validities arising from the assumption that the worlds share a common temporal ordering. Formulas (2) and (3) are two such validities, corresponding to the principle that one world has a first moment if and only if all worlds do.

$$(2) \quad P[\varphi \vee \neg\varphi] \leftrightarrow \Box P[\varphi \vee \neg\varphi]$$

$$(3) \quad H[\varphi \wedge \neg\varphi] \leftrightarrow \Box H[\varphi \wedge \neg\varphi]$$

In case $\approx_t$ is the universal relation for every t (or the relations $\approx_t$ are simply omitted from the satisfaction conditions) there are other validities, such as (4) and (5).

$$(4) \quad P\Box\varphi \rightarrow \Box P\varphi$$

$$(5) \quad F\Box\varphi \rightarrow \Box F\varphi$$

As far as I know, the general problem of axiomatising these logics has not been solved. But I'm not sure that it is worth doing, except as an exercise. The completeness proofs should not be difficult, using Gabbay's techniques (described in Section 4, below). and these logics do not seem particularly interesting from a philosophical point of view.

But a more interesting case is near to hand. The tendency we have noted to bring Carnap's metalinguistic notion of necessity down to earth has made room for the reintroduction of one of the most important notions of necessity: practical necessity, or historical necessity.⁷ This is the sort of necessity that figures in Aristotle's discussion of the Sea Battle (*De Int.* 18^b25–19^b4), and that arises when free will is debated. It also seems to be an important background notion in practical reasoning. Jonathan Edwards, in his usual lucid way, gives a very clear statement of the matter.

Philosophical necessity is really nothing else than the full and fixed connection between the things signified by the subject and predicate of a proposition, which affirms something to be true [This connection] may be fixed and made certain, because the existence of that thing is already come to pass; and either

⁷I am not sure if there are personal, or relational varieties of inevitability; it seems a bit peculiar to my ear to speak of an accident John caused as inevitable for Mary, but not inevitable for John. If there are such sorts of inevitability I mean to exclude them, and to speak only of impersonal inevitability. Thus 'inevitable' does not belong to the same modal family as 'able', since the latter is personal.

now is, or has been; and so has as it were made sure of existence. And therefore, the proposition which affirms present and past existence of it, may be this means be made certain, and necessarily and unalterably true; the past event has fixed and decided that matter, as to its existence; and has made it impossible but that existence should be truly predicted of it. Thus the existence of whatever is already come to pass, is not become necessity; 'tis become impossible it should be otherwise than true, that such a thing has been. [Edwards, 1957, pp. 152–3]

Historical necessity can be fitted into the $T \times W$ framework; it is merely a matter of adjusting the relations $\approx_t$ so that if $w \approx_t w'$, then w and w' share the same past up to and including t . So for $t' < t$, atomic formulas must be treated the same way in w and w' . furthermore, we have to stipulate that historical possibilities diminish monotonically with the passage of time: if $t < t'$, then $\{w' : w \approx_{t'} w'\} \subseteq \{w' : w \approx_t w'\}$. This interaction between time and relative necessity creates distinctive validities, such as (6) and (7).

$$(6) \quad \varphi \leftrightarrow \Box\varphi, \text{ if } \varphi \text{ contains no occurrences of } F.$$

$$(7) \quad P\Box\varphi \rightarrow \Box P\varphi$$

Formula (8), on the other hand, is clearly invalid.

$$(8) \quad \Box Pp \rightarrow P\Box p$$

These correspond to rather natural intuitions relating the flow of time to the loss of possibilities.

There is another way of representing historical necessity, which perhaps will seem less straightforward to logicians steeped in possible worlds. Time can be treated as non-linear (branching only towards the future), and worlds represented as branches on the resulting ordered structure. This corresponds very closely to the $T \times W$ account: (6) and (7) remain valid, and (8) invalid. But the validities are not the same. This matter will be taken up below, in Section 4.

So much for necessity; I will deal more briefly with 'ought' and conditionals.

As Aristotle points out, we don't deliberate about just anything; in particular, we deliberate only about what is in our power to determine. [Ne 1112^a 19f.] But the past, and the instantaneous present, are not in our power: deliberation is confined to future alternatives.

This suggests that deontic logic, insofar as it investigates practical oughts, should identify its possibilities with the ones of historical necessity. Unfortunately, this conception played little or no role in the early interpretation of deontic logic; those who developed the deontic applications of possible worlds semantics seemed to think of deontic possibilities ahistorically, as

‘perfect worlds’ in which all norms are fulfilled.⁸ Historical possibilities, on the other hand, are typically imperfect; life is full of occasions on which we have to make the best of a bad situation. In my opinion, this is one reason why deontic logic has seemed to most philosophers to consist largely of a sterile assortment of paradoxes, and why its influence on moral philosophy has been so fruitless.

Conditionals have been intensively studied by philosophical logicians over the last fifteen years, and this has created an extensive literature. Relatively little of this effort has been devoted to the interaction of conditionals with tense. But there is reason to think that important insights may be lost if conditionals are studied ahistorically. One very common sort of conditional (the philosopher’s novel example of a ‘subjunctive conditional’) is exemplified by (9).

(9) If Oswald hadn’t shot Kennedy, then Kennedy would be alive today.

These conditionals seem to be closely related to historical possibilities; they envisage courses of events that diverge at some point in the past from the actual one. And this in turn suggests that there may be close connections between historical necessity and some conditionals.

Examples like the following four provide evidence of a different sort.

(10) He would go if she would go.

(11) He will go if she will go.

(12) he would have gone if she were to have gone.

(13) he went if she went.

Sentences (10) and (11) seem hardly to differ in meaning, if (109) has to do with the future. On the other hand, (12) and (13) are very different. If he didn’t go, but would have gone if she had, (12) is true and (13) false.

This suggests that there may be systematic connections between tense, mood and the truth conditions of conditionals. According to one extreme proposal, the difference in ‘mood’ between (9) and the past-present conditionals form ‘If Oswald didn’t shoot Kennedy, then Kennedy is alive today’ can be accounted for solely in terms of the interaction of tense operators and the conditional.⁹ This has in favour of it the grammatical fact that ‘would’ is the past tense of ‘will’. But the matter is complex, and it is difficult to see how much merit there is in the suggestion.

There have been recent signs of interest in the interaction of tense and conditionals; the most systematic of these is [Thomason and Gupta, 1981]

⁸See [Von Wright, 1968] and Hintikka [1969; 1971].

⁹See [Thomason and Gupta, 1981, pp. 304–305].

If this study is any indication, the topic is surprisingly complicated. But the complications may prove to be of philosophical interest.

The relation between historical necessity and quantum mechanics is a topic that I will not discuss at any great length. The indeterminacy that is associated with microphenomena seems at first glance to invite a treatment using alternative futures; and one of the approaches to the measurement problem in quantum theory, the ‘many-worlds interpretation’, does appear to do just this. (See [DeWitt and Graham, 1973] for more information about the approach.)

But alternative futures don’t provide in themselves an adequate representation of the physical situation, because the quantum mechanical probabilities can’t be treated as distributions over a set of fully determinate worlds.¹⁰ Some further apparatus would have to be introduced to secure the right system of nonboolean probabilities, and as far as I can see, what is required would have to go beyond the resources of possible worlds semantics: there is no escaping an analysis of measurement interactions, or of interactions in general.

Possible world semantics may help to make the ‘many worlds’ approach to quantum indeterminacy seem less frothy; the prose of philosophical modal realists, such as D. Lewis [1970], is much more judicious than that in which the physicists sometimes indulge. (See, for instance, [DeWitt, 1973, p. 161].) So, modal logic may be of some help in sorting out the philosophical issues; but this leaves the fundamental problems untouched. Possible worlds are not in themselves a key to the problem of measurement in quantum mechanics.

The following sections will aim at fleshing out this general introduction with further historical information, more detailed descriptions of the relevant logical theories, and more extensive references to the literature.

3 HISTORICAL NECESSITY

The first sustained discussion of this topic, from the standpoint of modern tense logic is (as far as I know) Chapter 7 of [Prior, 1967] entitled ‘Time and determinism’.¹¹ Prior’s judgement and philosophical depth, as well as his

¹⁰See, for instance [Wigner, 1971] and [Fine, 1982].

¹¹The mention of historical necessity and its combination with deontic operators in the *tour de force* at the end of [Montague, 1968] probably takes precedence if you go by date of composition. But Montague’s discussion is very compressed, and neglects philosophical motivation. And some interesting things are said about indeterminism in Prior’s earlier book [Prior, 1957]. But the connection does not seem to be made there between the philosophical issues and the problem of interpreting future tense in treelike frames. The ingredients of a model theoretic treatment of historical necessity also occurred at an early date to Dana Scott. Like much of his work in modal and tense logic, it remained unpublished, but there is a mimeographed paper [Scott, 1967].

readable style, make this required reading for anyone seriously interested in historical necessity.

Prior's exposition is informal, and sprinkled with historical references to the philosophical debate over determinism. In this debate he unearths a logical determinist argument, that probably goes back to ancient times. According to this argument, if φ is true then, at any previous time, $F\varphi$ must have been true. But choose such a time, and suppose that at this earlier time φ could have failed to come about; then $F\varphi$ could not have been true at this time. It seems to follow that the determinist principle

$$(14) \quad \varphi \rightarrow H\Box F\varphi$$

holds good.

In discussing the argument and some ways of escaping from it, Prior is fairly flexible about this object language; in particular, he allows metric tense operators. Since these complicate matters from a semantic point of view, I will ignore them, and consider languages whose only modal operators are $\Box$ and the nonmetric tenses.

Also (and this is more unfortunate), Prior [1967] speaks loosely in describing models. At the place where his indeterminist models are introduced, for instance, he writes as follows.

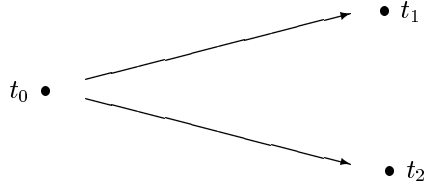
... we may define an Ockhamist *model* as a line without beginning or end which may break up into branches as it moves from left to right (i.e. from past to future), though not the other way
 ... (p. 126)

From this description it is clear that Prior is representing historical necessity by means of non-linear time, rather than according to the $T \times W$ format described in Section 2, above. But it is a little difficult to tell exactly what mathematical structures have been characterised; probably, Prior had in mind trees whose branches all have the order type of the (negative and non-negative) integers.

To bring this into accord with the usual treatment of linear nonmetric tense logic, we will liberalise Prior's account.

DEFINITION 1. A treelike frame $\mathfrak{A}$ for tense logic is a pair $\langle T, < \rangle$, where T is a non-empty set and $<$ is a transitive ordering on T such that if $t_1 < t$ and $t_2 < t$ then either $t_1 = t_2$ or $t_1 < t_2$ or $t_2 < t_1$.

As in ordinary tense logic, we imagine an assignment of truth values to atomic formulas at each $t \in T$, and truth-functional connectives are treated in the usual way. But things become perplexing when you try to interpret future tense in these structures. Take a very simple branching case, with just three moments, and imagine that p is true at t_0 and t_1 , and false at t_2 .



Is Fp true at t_0 ? It is hard to say.

Moreover, as you reflect on the problem, it becomes clear that Prior's juxtaposition of this technical problem with bits from figures like Diodorus Cronus, Peter de Rivo, and Jonathan Edwards is not merely an antiquarian quirk. There is a genuine connection. These treelike frames represent ways in which things can evolve indeterministically. A definition of satisfaction for a language with tense operators that is suited to such structures would automatically provide a way of making tense compatible with indeterministic cases. And it is just this that the logical argument for determinism claims can't be done. The technical problem can't be solved without getting to the bottom of this argument.

If the argument is correct, any definition of satisfaction for these structures will be incorrect—will generate validities that are at variance with the intended interpretation. Łukasiewicz's [1967] earlier three-valued solution is like this, I believe. Not because it makes some formulas neither true nor false, but because the formulas it endorses as valid are so far off the mark. It is bad enough that $Fp \vee \neg Fp$ is invalid, but also the approach would make $[[\Diamond Fp \wedge \Diamond \neg Fp] \wedge [\Diamond Fq \wedge \Diamond \neg Fq]] \rightarrow [Fp \leftrightarrow Fq]$ valid, if φ is true if and only if φ takes the intermediate truth value.¹²

Nor does the logic that Prior calls 'Peircian' strike me as more satisfactory, from a philosophical standpoint, though it does lead to some interesting technical problems relating to axiomatisability. Here, $F_{\text{varphi}}hi$ is treated as true at t in case the moments at which φ is true bar the future paths through t ; i.e., every branch through t contains a moment subsequent to t at which φ is true. On the Peircian approach, $F\varphi \vee \neg F\varphi$ is valid, but $Fp \vee F\neg p$ is not; nor is $p \rightarrow PFp$.¹³ As Prior says, sense can be made of this by reading F as 'will inevitably'. Though this helps us to see what is going on, it is not the intended interpretation.

The most promising of Prior's suggestions for dealing with indeterminist future tense is the one he calls 'Ockhamist'. The theory will be easier to present if we first work out the satisfaction conditions for $\Box\varphi$ in treelike frames. Intuitively, $\Box\varphi$ is true at t if φ is true at t no matter what the

¹²Prior briefly criticises Łukasiewicz's treatment of future tense [Prior, 1967, p. 135]; for a more extended criticism, see [Seeskin, 1971].

¹³Notice that this corresponds to one of the informal principles used in the logical argument for determinism.

future is like. And a way the future can be like will be represented by a fully determinate—i.e. linear—path beyond t . Since the frames are treelike, these correspond to the branches, or maximal chains, through t .

DEFINITION 2. Where $\langle T, < \rangle$ is a treelike model structure and $t \in T$, a branch through t is a maximal linearly ordered subset of T containing t ; B_t .

To make sense of φ being true at t no matter what the future is like, we will have to think of formulas being satisfied not just at moments t , but at pairs $\langle t, b \rangle$, where $b \in B_t$. Prior explains it this way. On the Ockhamist approach formulas like Fp are given ‘*prima facie* assignments’ at t ; such an assignment is made by choosing a particular b in B_t . His idea seems to be that there is something more provisional about the selection of b than about that of t ; but this he does not articulate or defend very fully. In the technical formulation of the theory there is no asymmetry between moments and branches; it is just that two parameters need to be fixed in evaluating formulas.

Once satisfaction is made relative to pairs $\langle t, b \rangle$ for some formulas, it must be relativised in the same way for all formulas; otherwise the recursive definition of satisfaction will become snarled. So, except for future tense, the definition will go like this.

DEFINITION 3. A function h assigning each atomic formula a subset of T is called an (Ockhamist) *assignment*, as in Chapter 1 of this volume.

DEFINITION 4. The h truth value $\|\varphi\|_{\langle t, b \rangle}^h$ of φ at the pair $\langle t, b \rangle$ is defined as follows. We assume here that $\|\varphi\|_{\langle t, b \rangle}^h = 0$ iff $\|\varphi\|_{\langle t, b \rangle}^h \neq 1$.

$$\begin{aligned} \|\varphi\|_{\langle t, b \rangle}^h &= 1 \text{ iff } t \in h(\varphi), \text{ if } \varphi \text{ is atomic,} \\ \|\neg\varphi\|_{\langle t, b \rangle}^h &= 1 \text{ iff } \|\varphi\|_{\langle t, b \rangle}^h = 0, \\ \|\varphi \wedge \psi\|_{\langle t, b \rangle}^h &= 1 \text{ iff } \|\varphi\|_{\langle t, b \rangle}^h = 1 \text{ and } \|\psi\|_{\langle t, b \rangle}^h = 1, \\ \|\varphi \vee \psi\|_{\langle t, b \rangle}^h &= 1 \text{ iff } \|\varphi\|_{\langle t, b \rangle}^h = 1 \text{ or } \|\psi\|_{\langle t, b \rangle}^h = 1, \\ \|\varphi \rightarrow \psi\|_{\langle t, b \rangle}^h &= 1 \text{ iff } \|\varphi\|_{\langle t, b \rangle}^h = 0 \text{ or } \|\psi\|_{\langle t, b \rangle}^h = 1, \\ \|P\varphi\|_{\langle t, b \rangle}^h &= 1 \text{ iff for some } t' < t, \|\varphi\|_{\langle t', b \rangle}^h = 1, \\ \|\Box\varphi\|_{\langle t, b \rangle}^h &= 1 \text{ iff for all } b \in B_t, \|\varphi\|_{\langle t, b \rangle}^h = 1. \end{aligned}$$

This definition renders $p \rightarrow \Box p$ valid, though not every substitution instance of it is valid. This can be easily changed by letting assignments take atomic formulas into subsets of $\{\langle t, b \rangle : t \in T \text{ and } b \in B_t\}$; [Prior, 1967, pp. 123–123] discusses the matter.

At this point, the way to handle $F\varphi$ is forced on us. We use the branch that is provided by the index.

$$\|F\varphi\|_{\langle t, b \rangle}^h = 1 \text{ iff for some } t' \in b \text{ such that } t < t', \|\varphi\|_{\langle t', b \rangle}^h = 1.$$

The Ockhamist logic is conservative; it's easy to show that if φ contains no occurrences of $\Box$ then φ is valid for treelike frames if and only if φ is valid in ordinary tense logic. So indeterminist frames can be accommodated without sacrificing any orthodox validities. This is good for those who (like me) are not determinists, but feel that these validities are intuitively plausible. Finally, the Ockhamist solution thwarts the logical argument for determinism by denying that if φ is true at t (i.e. at $\langle b, t \rangle$, for some selected b in B_t) then $\Box\varphi$ is.

This way out of the argument bears down on its weakest joint; but the argument is so powerful that even this link resists the pressure; it is hard for an indeterminist to deny that $\Box\varphi$ must be true if φ is. To a thoroughgoing indeterminist, the choice of a branch b through t has to be entirely *prima facie*; there is no special branch that deserves to be called the 'actual' future through t .¹⁴ Consider two different branches b_1 and b_2 , through t , with $t < t_1 \in b_1$ and $t < t_2 \in b_2$. From the standpoint of t_1 , b_1 is actual (at least up to t_1). From the standpoint of t_2 , b_2 is actual (at least up to t_2). And neither standpoint is correct in any absolute sense. In exactly the same way, no particular moment of linear time is 'present'.

But then it seems that the Ockhamist theory gives no account of truth relative to a moment t , and it also suggests very strongly that if φ is true at t then $\Box\varphi$ is also true at t . The only way that a thing can be true at a moment is for it to be settled at that moment.

In Thomason [1970], it is suggested that such an absolute notion of truth can be introduced by superimposing Van Fraassen's treatment of truth-value gaps onto Prior's Ockhamist theory.¹⁵ The resulting definition is very simple.

DEFINITION 5.

$$\begin{aligned} \|\varphi\|_t^h = 1 & \quad \text{iff} \quad \|\varphi\|_{\langle t, b \rangle}^h = 1 \quad \text{for all } b \in B_t, \\ \|\varphi\|_t^h = 0 & \quad \text{iff} \quad \|\varphi\|_{\langle t, b \rangle}^h = 0 \quad \text{for all } b \in B_t. \end{aligned}$$

This logic preserves the validities of linear tense logic; indeed, φ is Ockhamist valid if and only if it is valid here. Also, the rule holds good that if $\|\varphi\|_t^h = 1$ then $\|\Box\varphi\|_t^h = 1$.

Thus, this theory endorses the principle (rejected by the Ockhamist theory) that if a thing is true at t then its truth at t is settled. It may seem at first that it validates all the principles needed for the logical determinist argument, but of course (since the logic allows branching frames) it must sever the argument somewhere. the way in which this is done is subtle. The scheme $\varphi \rightarrow HF\varphi$ is valid, but this does not mean that if φ is true at t then $F\varphi$ is true at any $t' < t$. That is, it is *not* the case that if $\|\varphi\|_t^h = 1$ then

¹⁴See [Lewis, 1970], and substitute 'the actual future' for 'the actual world' in what he says. *That* is the view of the thoroughgoing indeterminist.

¹⁵See, for instance Van Fraassen [1966; 1971].

$\|F\varphi\|_{t'}^h = 1$ for all $t' < t$. The validity of this scheme only means that for all $b \in B_t$, if $\|\varphi\|_{\langle t, b \rangle}^h = 1$ then $\|F\varphi\|_{\langle t', b \rangle}^h = 1$ for all $t' < t$. But there may be $b' \in B_{t'}$ which are not in B_t .

To put it another way, the fact that $F\varphi$ is true at t from the perspective of a later t' does not make $F\varphi$ *absolutely* true at t , and so need not imply that $\Box F\varphi$ is true at t .

This manoeuvre makes use of the availability of truth-value gaps. To make this clearer, take a future-oriented version of the logical determinist argument: $F\varphi \vee F\neg\varphi$ is true at any t ; so $F\varphi$ is true at t or $\neg F\varphi$ is true at t ; so $\Box F\varphi$ is true at t or $\Box \neg F\varphi$ is true at t . The supervaluational theory blocks the second step of this argument: in any such theory, the truth of $\psi \vee \neg\psi$ does not imply that ψ is true or $\neg\psi$ is true.

Thomason suggests that this logic represents the position endorsed by Aristotle in *De Int.* 18^b25 – 19^b4, but his suggestion is made without any analysis of the very controversial text, or discussion of the exegetical literature. For a close examination of the texts, with illuminating philosophical discussion, see [Frede, 1970]; see also [Sorabji, 1980]. for a broadly-based examination of Aristotle's views that nicely illustrates the value of treelike frames as an interpretive device, see [Code, 1976]. The suggestion is also made in [Jeffrey, 1979]. For information about the medieval debate on this topic, see [Normore, 1982].

4 THE TECHNICAL SIDE OF HISTORICAL NECESSITY

The mathematical dimension of the picture painted in the above section is still relatively undeveloped. At present, most of the results known to me deal with axiomatisability in the propositional case.

We have already characterised one important variety of propositional validity: φ is *Ockhamist valid* if it is satisfied at all pairs $\langle t, b \rangle$, relative to all Ockhamist assignments on all treelike frames. (See Definitions 2–4, above.) The time has come to give an official definition of $T \times W$ validity.

DEFINITION 6. A $T \times W$ frame is a quadruple $\langle W, T, <, \approx \rangle$, where W and T are non-empty sets, $<$ is a transitive relation on T which is also irreflexive and linear (i.e. $t \not< t$ for all $t \in T$, and either $t < t'$ or $t' < t$ or $t = t'$ for all, $t, t' \in T$), and $\approx$ is a 3-place relation on $T \times W \times W$, such that (1) for all t , $\approx_t$ is an equivalence relation (i.e. $w \approx_t w$ for all $t \in T$ and $2 \in W$, etc.), and (2) for all $w_1, w_2 \in W$ and $t, t' \in T$, if $w_1 \approx_t w_2$ and $t' < t$ then $w_1 \approx_{t'} w_2$. The intention is that $w \approx_t w'$ if w and w' are historical alternatives through t , and so differ only in what is future to t .

DEFINITION 7. A function h assigning each atomic formula a subset of $T \times W$ is an *assignment*, provided that if $w \approx_t w'$ and $t_1 \leq t$ then $\langle t_1, w \rangle \in h(\varphi)$ iff $\langle t_1, w' \rangle \in h(\varphi)$.

DEFINITION 8. The h -truth value $\|\varphi\|_{\langle t, w \rangle}^h$ of φ at the pair $\langle t, w \rangle$ is defined by a recursion that treats truth-functional connectives in the usual way. The clauses for tense and necessity run as follows.

$$\begin{aligned} \|P\varphi\|_{\langle t, w \rangle}^h &= 1 \quad \text{iff for some } t' < t, \|\varphi\|_{\langle t', w \rangle}^h = 1, \\ \|F\varphi\|_{\langle t, w \rangle}^h &= 1 \quad \text{iff for some } t' \text{ such that } t < t', \|\varphi\|_{\langle t', w \rangle}^h = 1, \\ \|\Box\varphi\|_{\langle t, w \rangle}^h &= 1 \quad \text{iff for all } w' \text{ such that } w \approx_t w', \|\varphi\|_{\langle t, w' \rangle}^h = 1. \end{aligned}$$

A formula is $T \times W$ valid if it is satisfied at every pair $\langle t, w \rangle$ by every assignment on every $T \times W$ frame.¹⁶

There are some validities that are peculiar to these $T \times W$ frames, and that arise from the fact that only a single temporal ordering is involved in these frames: (15) and (16) are examples,

$$(15) \quad FG[p \wedge \neg p] \rightarrow \Box FG[p \wedge \neg p]$$

$$(16) \quad GF[p \vee \neg p] \rightarrow \Box GF[p \vee \neg p]$$

Example (15) is valid because its antecedent is true at $\langle t, w \rangle$ if and only if there is a t' that is $<$ -maximal with respect to w ; but this holds if and only if there is a t' that is $<$ -maximal absolutely. Example (16) is similar, except that this time what is at stake is the non-existence of a maximal time.

Burgess remarked (in correspondence) that $T \times W$ validity is recursively axiomatisable, since it is essentially first-order. But as far as I know the problem of finding a reasonable axiomatisation for $T \times W$ validity is open. I would expect the techniques discussed below, in connection with Kamp validity, to yield such an axiomatisation.

Although (15) and (16) may be reasonable given certain physical assumptions, they do not seem so plausible from a logical perspective. After all, if $w \approx_t w'$, all that is required is that w and w' should share a certain segment of the past, and this implies that the structure of time should be the same in w and w' on this segment. But it is not so clear that w and w' should participate in the same temporal structure after t . This suggests a more liberal sort of $T \times W$ frame, first characterised by Kamp [1979].¹⁷

¹⁶I will adhere to this terminology here, but I am not confident that it is the best terminology, over the long run. Varieties of $T \times W$ validity tend to proliferate, and this is only one of them, and probably not the most interesting. Perhaps it would be better to speak of (Fixed T) $\times W$ validity— but this is awkward.

¹⁷This paper of Kamp's deserves summary because it became widely known and is historically important, though it was never published. Kamp had evidently been thinking about these matters for several years; the latest draft of the paper that I have seen was finished early in 1979. The paper contains much valuable philosophical discussion of historical necessity, and defines the type of validity that I here call 'Kamp validity'. An axiomatisation is proposed in the paper, which was never published because of difficulties that came to light in the completeness argument that Kamp had sketched for validity in dense Kamp frames. In 1979, Kamp discovered a formula that was Kamp valid but

DEFINITION 9. A Kamp frame is a triple $\langle \mathcal{T}, W, \approx \rangle$ where W is a non-empty set, $\mathcal{T}$ is a function from W to transitive, irreflexive linear orderings (i.e. if $w \in W$ then $\mathcal{T}(w) = \langle T_w, <_w \rangle$, where $<_w$ is an ordering on T_w as in $T \times W$ frames), and $\approx$ is a relation on $\{ \langle t, w, w' \rangle : w, w' \in W \text{ and } t \in \mathcal{T}(w) \cap \mathcal{T}(w') \}$ such that for all $t, \approx_t$ is an equivalence relation, and if $w \approx_t w'$ then $\{t_1 : t_1 \in T_w \text{ and } t_1 <_w t\} = \{t_1 : t_1 \in T_{w'} \text{ and } t + 1 <_{w'} t\}$. Also, if $w \approx_t w'$ and $t' <_w t$ then $w \approx_{t'} w'$.

The definitions of an assignment, and of the h -truth value $\|\varphi\|_{\langle t, w \rangle}^h$ of φ at the pair $\langle t, w \rangle$ (where $t \in \mathcal{T}_w$ and h is an assignment) are readily adapted from Definitions 7 and 8.

Besides (AK0) all classical tautologies, Kamp takes as axioms all instances of the following schemes.¹⁸

$$(AK1) \quad H[\varphi \rightarrow \psi] \rightarrow [P\varphi \rightarrow P\psi]$$

$$(AK2) \quad G[\varphi \rightarrow \psi] \rightarrow [F\varphi \rightarrow F\psi]$$

$$(AK3) \quad PP\varphi \rightarrow P\varphi$$

$$(AK4) \quad FF\varphi \rightarrow F\varphi$$

$$(AK5) \quad PG\varphi \rightarrow \varphi$$

$$(AK6) \quad FH\varphi \rightarrow \varphi$$

$$(AK7) \quad PF\varphi \rightarrow [P\varphi \vee \varphi \vee F\varphi]$$

$$(AK8) \quad FP\varphi \rightarrow [P\varphi \vee \varphi \vee F\varphi]$$

$$(AK9) \quad \Box[\varphi \rightarrow \psi] \rightarrow [\Box\varphi \rightarrow \Box\psi]$$

$$(AK10) \quad \Box\varphi \rightarrow \varphi$$

$$(AK11) \quad \Diamond\varphi \rightarrow \Box\Diamond\varphi$$

$$(AK12) \quad \Diamond P\varphi \rightarrow P\Diamond\varphi$$

$$(AK13) \quad \Box\varphi \vee \Box\neg\varphi, \text{ if } \varphi \text{ is atomic.}$$

not provable from the axioms of the paper. Later, Thomason discovered other sorts of counterexamples. As far as I know, the axiomatisation problem for Kamp validity was open until Dov Gabbay encountered the problem at a workshop for [the first edition of] this *Handbook*, in the fall of 1981. Due to the wide circulation of [Kamp, 1979], a number of erroneous references have crept into the literature concerning the existence of an axiomatisation of Kamp validity. Gabbay has not yet published his result [Editors' note: see Section 7.7 of D. M. Gabbay, I. Hodkinson and M. Reynolds. *Temporal Logic*, volume 1, Oxford University Press, 1994.], and any such reference in a work published before [1981, the first edition of] this *Handbook* is likely to be mistaken.

¹⁸I omit the axiom scheme for density, and omit a redundant axiom for $\Box$. And the system I describe is only one of several discussed by Kamp.

As well as (RK0) *modus ponens*, Kamp posits the rules given by the following three schemes.

$$(RK1) \quad \varphi / H\varphi$$

$$(RK2) \quad \varphi / G\varphi$$

$$(RK3) \quad \varphi / \Box\varphi$$

Readers familiar with axioms for modal and tense logic will see that this list falls into three natural parts. Classical tautologies, *modus ponens*, (RK1), (RK2) and (AK1)–(AK8) are familiar principles of ordinary tense logic without modality. Classical tautologies, *modus ponens*, (RK3), and (AK9)–(AK11) are principles of the modal logic **S5**. Axiom (AK12) is a principle combining tense and modality; this principle was explained informally in Section 2. The validity of (AK13) reflects the treatment of atomic formulas as noncontingent; see the provision in Definition 7. If tense operators were not present, (AK13) would of course trivialise $\Box$, rendering every formula noncontingent.

To establish the incompleteness of (AK0)–(AK13) + (RK0)–(RK3), consider the formula (17), discovered by Kamp, where $E_1(\varphi)$ is $F\varphi \wedge G[\varphi \vee P\varphi]$.

$$(17) \quad [PE_1(p) \wedge \Diamond PE_1(q)] \rightarrow [P[E_1(p) \wedge P\Diamond E_1(q)] \vee \vee P[\Diamond E_1(q) \wedge \Diamond E_1(p)] \vee P[E_1(p) \wedge \Diamond E_1(q)]].$$

The validity of (17) in Kamp frames follows from the fact that these frames are closed under the sort of diagram completion given in Figure 1. Given $t_1, t_2, t_3, t'_1, t'_3$ and the relations of the diagram, it must be possible to interpolate at t'_2 in w' alternative to t_2 , with $t'_3 < t'_2 < t'_1$.¹⁹ Formula (17) is complicated, but I think I can safely leave the task to checking its Kamp validity to the reader.

One proof that (17) is independent of (AK0)–(AK13) + (RK0)–(RK3) makes use of still another sort of frame, which is closer to a Henkin construction than the $T \times W$ frames. If our task is to build models out of maximal consistent sets of formulas, the ‘times’ of a $T \times W$ frame are rather artificial; they would have to be equivalence sets of maximal consistent sets. In *neutral frames*, the basic elements are moments, or instantaneous slices of evolving worlds, which are organised by intra-world temporal relations, and interworld alternativeness relations. Neutral frames tend to proliferate, because of the many conditions that can be imposed on the relations; hence my use of subscripts in describing them.

DEFINITION 10. A neutral frame₁ is a triple $\langle W, \mathcal{U}, \approx \rangle$, where (1) W is a nonempty set, (2) $\mathcal{U}$ is a function whose arguments are members of w of W

¹⁹In fact, since we are working with Kamp frames, $t'_2 = t_2$. But I put it in this more general way in anticipation of what I will call *neutral* frames, so that Figure 1 will be similar in format to Figure 2.



Figure 1. Interpolation

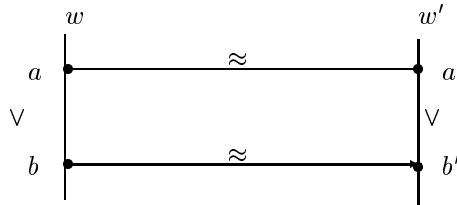


Figure 2. One-way completion.

and whose values are orderings $\langle U_w, <_w \rangle$ such that $U - w$ is a nonempty set and $<_w$ is a transitive²⁰ ordering on U_w such that for all $a, b \in U_w$ either $a <_w b$ or $b <_w a$ or $a = b$, (3) if $w, w' \in W$ and $w \neq w'$ then $\mathcal{U}_w$ and $\mathcal{U}_{w'}$ are disjoint, and (4) $\approx$ is an equivalence relation on $\cup\{U_w : w \in W\}$, and (5) if $a \approx a'$ and $b <_w a$ then there is some $b' \in \mathcal{U}_{w'}$ (where $a' \in \mathcal{U}_{w'}$) such that $b \approx b'$ and $b' <_{w'} a'$.

Here, each U_w corresponds to the set of instantaneous slices of the world w ; $\approx$ is the alternativeness relation between these slices.

The diagram-completion property expressed in (5) looks as shown in this picture. It's important to realise that nothing prevents $a \neq b$ while at the same time $a' = b'$ in Figure 2. Of course, in this case we will also have $a \approx b$, which would be something like history repeating itself, at least in all respects that are settled by the past.

Everything generated by (AK0)–(AK13) + (RK0)–(RK3) is valid in neutral frames₁. (And the fact that the converse holds, so that we have a completeness result; see [Thomason, 1981c].) But it is easy to show that (17) is invalid in neutral frames₁.

²⁰But not necessarily irreflexive!

This process can be continued; for instance, stronger conditions of diagram completion can be imposed on neutral frames, which extend the propositional validities, and completeness conditions obtained for the resulting sorts of frames. Details can be found in [Thomason, 1981c]; but this effort did not produce an axiomatisation of Kamp validity.

Using his method of constructing irreflexive models (see [Gabbay, 1981]), Gabbay has shown that all the Kamp validities will be obtained if (AG1) and (RG1) are added to (AK0)–(AK13) + (RK0)–(RK3). Some terminology is needed to formulate (RG1); we need a way of talking, to put it intuitively, about formulas which record a finite number of steps forwards, backwards and sideways in Kamp frames.

DEFINITION 11. Let $\alpha_i \in \{P, F, \Diamond\}$ for $1 \leq i \leq n, n \geq 0$. Then $f(\varphi) = \varphi$, and

$$f_{\alpha_1, \dots, \alpha_n}(\varphi_0, \dots, \varphi_{n-1}, \varphi_n) = \varphi_0 \wedge \alpha_1[\varphi_1 \wedge \dots \wedge \alpha_{n-1}[\varphi_{n-1} \wedge \alpha_n \varphi_n] \dots].$$

So, for instance, $f_{F, \Diamond, P}(p_0, p_1, p_2, p_3)$ is $p_0 \wedge F[p_1 \wedge \Diamond[p_2 \wedge Pp_3]]$.

The axiom and rule are as follows:

$$(AG1) \quad [\Box \neg \varphi \wedge H\Box \varphi \wedge \Box \psi] \rightarrow G\Box H[[\neg \varphi \wedge H\varphi] \rightarrow \psi]$$

$$(RG2) \quad \frac{f_{\alpha_1, \dots, \alpha_n}(\varphi_0, \dots, \varphi_{n-1}, [\varphi_n \wedge \neg p \wedge Hp]) \rightarrow \psi}{f_{\alpha_1, \dots, \alpha_n}(\varphi_0, \dots, \varphi_{n-1}, \varphi_n) \rightarrow \psi}$$

In RG2, p must be foreign to ψ and $\varphi_0, \dots, \varphi_n$.

I leave it to the reader to verify that the axiom and rules are Kamp valid.

It looks as if the ordering properties of linear frames that can be axiomatised in ordinary tense logic (endlessness towards the past, endlessness towards the future, density, etc.)²¹ can be axiomatised against the background of Kamp frames, using Gabbay's techniques. But even so, there are still many simple questions that need to be settled; to take just two examples, it would be nice to know whether Kamp satisfaction is compact, and whether (RG2) is independent.

The situation with respect to treelike frames (i.e. with respect to Ockhamist validity) was even less well explored until recently. To begin with, a number of people have noticed that, although every propositional formula that is Kamp valid is Ockhamist valid,²² the converse fails. Nishimura [1979b] points this out, giving (18) as a counterexample.²³

$$(18) \quad \frac{GH\Box FFP[H\neg p \wedge \neg p \wedge Gp] \rightarrow FPD\Box FFP[\neg p \wedge \Box Gp]}{}$$

²¹See Chapter 1 of this Volume.

²²This follows from the fact that a Kamp model can be made from a treelike model, by making worlds out of its branches.

²³Kamp [1979] ascribes a similar example to Burgess.

- (21) Inevitably, life on earth will come to an end at some date in the future.
- (22) For every date in the future, it is not inevitable that life on earth will have come to an end by that date.

For this reason, it seems to me that the $T \times W$ frames do not have the philosophical interest of the treelike ones, though they are certainly interesting for technical reasons. This makes Ockhamist validity appear worth investigating; until recently, however, very little was known about it.²⁴ In [Burgess, 1979], it is claimed that Ockhamist validity is recursively axiomatisable, and a proof is sketched. Later (in conversation), Kripke challenged the proof and Burgess has been unable to substantiate all the details. Very recently, Gurevich and Shelah have proved a result implying that Ockhamist validity is decidable. (See [Gurevich and Shelah, 1985].) At present (October 1982) their paper is not yet written, and I have not had an opportunity to see their proof. The main result is that the theory of trees with second-order quantification over maximal chains is decidable.

Of course a proof of decidability would allow axioms to be recovered for Ockhamist validity; but this would be done in Craigian fashion. And unfortunately, Gabbay's completeness techniques do not seem (at first glance, anyway) to extend to the treelike case. Burgess' example [1979, p. 577], of an Ockhamist invalid formula valid in countable treelike frames, helps to bring home the complexity of the case.²⁵

There are some interesting technical results regarding logics other than the treelike and $T \times W$ ones that I have stressed here; the most important of these is Burgess' [1980] proof that the Peircian validities are decidable. Burgess has pointed out to me that the method of Section 5 of Burgess [1980] can be used to prove Kamp validity decidable, as well as $T \times W$ validity with dense time. He also remarks that the most interesting technical questions about the case in which atomic formulas are treated like complex ones (so that $p \rightarrow \Box p$ is not valid, and substitution is an admissible rule) are unresolved, and may prove more difficult.

5 DEONTIC LOGIC COMBINED WITH HISTORICAL NECESSITY

For a general discussion of deontic logic, with historical background, see Føllesdal and Hilpinen [1971] and Aqvist's chapter on Deontic Logic in Volume 8 of this *Handbook*. This presentation will concentrate on combinations

²⁴In note 15 of [Thomason, 1970], Thomason says that he 'means to present an axiomatisation' of Ockhamist validity in a forthcoming paper. Since the paper has never appeared, this intention was evidently premature.

²⁵I owe much of the information in this paragraph to Burgess and Gurevich.

of deontic modalities with temporal ones, and, in particular, with historical necessity.

Deontic logic seems to have suffered from a lack of communication. Even now, papers are written in which the relevant literature is not mentioned and the authors appear to be reinventing the wheel. In the hope that a survey of the literature will help to correct this situation, I have tried to make the bibliographical coverage of the present discussion thorough.

One facet of this lack of communication can be seen at work in the late 1960s. On the one hand, quite sophisticated model theoretic studies were developed during this time, treating deontic possibilities historically, as future alternatives. Montague [1968, pp. 116–117] and Scott [1967] represent the earliest such studies. (Unfortunately, Montague's presentation is tucked away in a rather forbidding technical paper that discusses many other topics, and the publication of the paper was delayed. And Scott's paper was never published.) But in 1969 Chellas [1969] appeared, giving an extended and very readable presentation of the California Theory.²⁶ (Montague's, Scott's and Chellas' theories are quite similar variations of the $T \times W$ approach; the treatment of historical necessity is similar, and indeed identical in all important respects to Kamp's.)

But although Chellas' monograph contains an extensive and valuable bibliography of deontic logic, including many references to the literature in moral philosophy and practical reasoning, and though Chellas is evidently familiar with this literature, there is no attempt in the work to relate the theory to the more general philosophical issues, or even to discuss its application to the 'paradoxes' of deontic logic, which by then were well known. Chellas concentrates on the mathematical portion of the task. Thus although the presentation is less compressed than Montague's it remains relatively impenetrable to most moral philosophers, and there is no advertisement of the genuine help that the theory can give in dealing with these puzzles.

On the other hand, in the philosophical literature, it is easy to find studies that would have benefited from vigorous contact with logical theories such as Chellas'. To consider an example almost at random, take [Chisholm, 1974]. This paper has the word 'logic' in its title and deals with a topic that is thoroughly entangled with Chellas' investigation, but Chisholm's paper has no references to such logical work and seems entirely ignorant of it. Moreover, the paper is written in an axiomatic style that makes no use of semantical techniques that had been current in the logical literature for many years. And Chisholm commits errors that could have been avoided by awareness of these things. ([Chisholm, 1974, D9 on p. 13] is an example;

²⁶Chellas' study is concerned with imperatives; but he starts with the assumption that these express obligations, so that the work belongs to deontic logic. Chellas [1969] is difficult to obtain; the theory is also presented in [Chellas, 1971], which is more accessible. McKinney [1975], a dissertation written under Chellas, is a later work belonging to this genre.

compare this with [Thomason, 1981a, pp. 183–184].)

To take another example, Wiggins [1973] provides an informal discussion, within the context of the determinist-libertarian debate in moral philosophy, of issues very similar to those treated by Chellas [1971] (and, so far as historical necessity goes in [Prior, 1967, Chapter 7]). Again, the paper contains no references to the logical literature. Though Wiggins' firm intuitive grasp of the issues prevents his argument from being affected,²⁷ it would have been nice to see him connect his account to the very relevant modal theoretic work.

There are a number of general discussions of the 'paradoxes' of deontic logic: see, for example, Åqvist [1967, pp. 364–373], Føllesdal and Hilpinen [1971, pp. 21–26], Hansson [1971, pp. 130–133], Al-Hibri [1978, pp. 22–29] and Van Eck [1981, pp. 28–35]. It should be apparent from the shudder quotes that I prefer not to dignify these puzzles with the same term that is applied to profoundly deep (perhaps unanswerable) questions like the Liar Paradox. The puzzles are a disparate assortment, and require a spectrum of solutions, some of which have little to do with tense. The Good Samaritan problem, for instance (the problem of reparational obligations), seems from one point of view to simply be a rediscovery of the frailties of the material conditional as a formalisation of natural language conditionals.

But part of the solution²⁸ of this problem seems to lie in the development of an *ought kinematics*,²⁹ in analogy to the probability kinematics that is the topic of [Jeffrey, 1990, Chapter 11]. As we would expect from probability, where there are interactions (surprisingly complex ones) between the rules of probability kinematics and locutions that combine conditionality and probability, we should expect there to be close relationships between ought kinematics and the semantics of conditional oughts. Nevertheless, we can formulate the kinematics of ought without having to work with conditional oughts in the object language.

²⁷The one exception is Wiggins' definition of 'deterministic theory' and his assumption that macroscopic physical theories are deterministic. Here his argument could have been genuinely improved by familiarity with Montague [1962]. I haven't discussed this much-neglected paper here, because it belongs more to philosophy of science than to modal logic.

²⁸Another part consists in developing an account of conditional oughts. Since this is a combination of the modalities rather than of tense and modality, I do not discuss it here. See De Cew [1981] for a recent survey of the topic.

²⁹I have coined this term, since there seems to be a terminological niche for it. The sources are Jeffrey's 'probability kinematics' (I believe the term is his) and Greenspan's use of 'ought' as a substantive. I think there are good reasons for the terminology she recommends in [Greenspan, 1975] where she speaks of 'having an ought'. A theory like Chellas' cannot, of course, be deployed without containing an ought kinematics; but, as I said, the California writers did not advertise the theory as a solution of some of the deontic puzzles. Powers [1967] juxtaposes the two, though his pay-off machines present the model theory informally. Thomason [1981b] states the modal theory in terms of treelike frames, and discusses it in relation to some of the deontic puzzles.

The technical resolution of this problem is very simple, and this is precisely what the California theory that I referred to above provides: e.g. the theory of Chellas [1969]. for the sake of variety, I will formulate it with respect to treelike frames; this is something that, to my knowledge, was first done in Thomason [1981b].³⁰

DEFINITION 12. A treelike frame $\mathfrak{A}$ for deliberative deontic tense logic is a pair $\langle T, <, O \rangle$, where $\langle T, < \rangle$ is a treelike frame for tense logic and O is a function on T such that for all $t \in T$, (1) O_t is nonempty, (2) $O_t \subseteq B_t$, and (3) if $t < t'$ and $t' \in b$ for some $b \in O_t$, then $O_{t'} = O_t \cap B_{t'}$.

The satisfaction clause for oughts is as follows.

$$\|O_\varphi\|_{\langle t, b \rangle}^h = 1 \text{ iff for all } b' = O_t, \quad \|\varphi\|_{\langle t, b' \rangle}^h = 1.$$

We are dealing with the result of extending a propositional language of the sort we discussed in previous sections (truth- functional connectives, past and future tense, and historical $\Box$) by adding a one-place connective O for ought.³¹

Something needs to be said about Condition (3) on frames, which is not to be found in Thomason [1981b], and, as far as I know, has also been overlooked by other authors who (like Chellas) formulate the model theory of ought kinematics. This condition yields validities such as the following two.

$$(23) \quad OG[F\varphi \rightarrow \neg OG\neg\varphi]$$

$$(24) \quad OG\varphi \rightarrow OGO\varphi$$

These do strike me as valid in this context, and at any rate are interesting tense-deontic principles having to do with the coherence of plans. Principle (23), for instance, disallows an alternative future in O_t along which some outcome will happen, but is forbidden from ever happening. Such an alternative can't correspond to a coherent plan.

This set-up can readily deal with reparational obligations. Suppose that, because of a promise to my aunt, at 4:00 I ought to catch an airplane at 5:00, but that at 5:00 I have broken my promise because of the attractions of the airport bar. Then at 5:00 I should call my aunt to tell her I won't be on the plane. Though this is the sort of situation that is sometimes represented as paradoxical in the literature, it is easily modelled in ought kinematics, with no apparent conceptual strain. At one time we have $O\neg Fp$ true, where p stands for 'I tell my aunt I won't be on the plane'. At a later time (one that involves the occurrence of something that shouldn't have happened) OFp is true.

³⁰The theory of Thomason [1981b] is more general, but yields an account of deliberative ought as a special case.

³¹Since ' P ' is already used for past tense, I will not use any special symbol for the dual of O .

If we press the account a bit harder, we can change the example; it also is true at 5:00 that I ought to inform my aunt I won't be on the plane, and this can be taken to entail that I ought not to be on the plane, since I can only inform someone of what is true, and because $\Box[\varphi \rightarrow \psi] \rightarrow [O\varphi \rightarrow O\psi]$ is a validity of our logic.

The response to this pressure is, of course, a Gricean manoeuvre;³² it is true at 5:00 (on the understanding of 'ought' in question) that I ought not to be on the plane. But it is not worth saying at 5:00, and if I were to say it then, I would be taken to have said something else, something false. And all of this can be made plausible in terms of general principles of reasonable conversation. I will not give the details here, since they can easily be reconstructed from [Lewis, 1979b].³³ On balance, the approach seems to be well braced against pressure from this direction. It is hard to imagine a reasonable theory of truth conditions that will not have to deploy Gricean tactics at some point. And this can be made to look like a very reasonable place to deploy them.

These linguistic reflections are nicely supported by quite independent philosophical considerations. Greenspan [1975] is a sustained study of ought kinematics from a philosophical standpoint, in which it is argued that a time-bound treatment of oughts is essential to an understanding of their logic, and that the proper view of deontic detachment is that a conditional ought licenses a 'consequent ought' when the antecedent is unalterably true. The paper contains many useful references to the philosophical literature, and provides a good example of the results that can be obtained by combining this philosophical material with the logical apparatus.

The fact that 'I ought to be on the plane' would ordinarily be taken to be false at 5:00 in the example we discussed above shows that 'ought' has employments that are not practical or deliberative: ones that perhaps have to do with wishful thinking. Also, there is its common use to express a kind of necessity: 'The butter is warm enough now; it ought to melt'.

Philosophers are inclined to speak of ambiguity in cases like this, but this is either a failure to appreciate the facts or an abuse of the word 'ambiguity'. The word 'ought' is indexical or context-sensitive, not ambiguous. The matter is argued, and some of its consequences are explored, in Kratzer [1977]; see [Lewis, 1979b] for a study of the consequences in a more general setting.

In Thomason [1981b] this is taken into account by a more general interpretation of *O*, according to which the relevant alternatives at *t* need not be possible futures for *t*. Probably the most general account would make the interpretation of *O* in a context relative to a set of alternatives which are regarded as possible in some sense relative to that context.

³²With an added wrinkle, due to the contextual adjustability of oughts.

³³See especially [Lewis, 1979b, pp. 354–355].

This is of course related to the philosophical debate over whether ‘ought’ implies ‘can’.³⁴ The most sophisticated linguistic account makes the issue appear rather boring: if we attend to a reasonable distinction between ambiguity and context-sensitivity, ‘ought’ doesn’t imply ‘can’, since there are contexts that provide counterexamples. But in practical contexts, when the one is true the other will be, even if for technical reasons we can’t relate this to an implication among linguistic forms.³⁵ This result is rather disappointing. But maybe there are ways of extracting interesting consequences for moral philosophy from a pragmatic account of oughts and other practical phenomena. An idea that I find intriguing is that manipulation of the context is the typical—perhaps the only—mechanism of moral weakness. The idea is suggested in [Thomason, 1981a], but is not much developed.

There seems to be no point in discussing the technical side of temporal deontic logic here. Not much work has been done in the area,³⁶ and the best strategy seems to be to let the matter wait until more is known about the interpretation of historical necessity.

Strictly speaking, conditional oughts are more closely related to the combination of conditionals with oughts than that of tense with modalities. Because the topic is complex and a thorough discussion of it would take up much space I have decided to neglect it in this article, even though (as Greenspan [1975] makes clear) tense enters into the matter. for an illuminating discussion of some of the problems, see DeCew [1981].³⁷

6 CONDITIONAL LOGIC COMBINED WITH HISTORICAL NECESSITY

In the present essay, ‘modality’ has been confined to what can be interpreted using possible worlds semantics. So here, ‘conditional logic’ has to do with the modern theories that were introduced by Stalnaker and then by D. Lewis; see [Lewis, 1973].³⁸ For surveys of this work, see the chapter on

³⁴See, for instance [Hare, 1963, Chapter 4].

³⁵The fate of the validity of ‘I exist’ seems to be much the same. Maybe linguistics is a graveyard for some philosophical slogans.

³⁶Some results are presented in [Åqvist and Hoepelman, 1981].

³⁷I have not discussed Castañeda’s work here, though it may offer an alternative to the kinematic approach to some of the deontic puzzles; see for instance [Castañeda, 1981]. for one thing, the work falls outside the topic of this article; for another, Castañeda’s writings on deontic logic strike me as too confused and poorly presented to repay close study.

³⁸Thus, for instance, I will not discuss [Slote, 1978], for although it deals with conditionals and the time it seeks to replace the possible worlds semantics with an analysis in the philosophical tradition. Another recent paper, [Kvart, 1980] uses techniques that are more model theoretic; but philosophical preconceptions have crept into the semantic theory at every point and uglified it. The usefulness of logical techniques in philosophy is largely dependent on the independence of the intuitions that guide the two disciplines; this enables them to reinforce one another in certain instances.

Conditional Logic by Nute and Cross in Volume 4 of this *Handbook*, and [Harper, 1981] and the volume in which it appears: Harper *et al.* [1981].

The interaction of conditionals and historical necessity is a topic that is only beginning to receive attention.³⁹ As in the case of deontic logic there is a fairly venerable philosophical tradition, involving issues that are still debated in the philosophical literature, and a certain amount of technical model theoretic work that may be relevant to these issues. But this time, the philosophical topic is causality.⁴⁰

A conditional like D. Lewis', which does not satisfy the principle of conditional excluded middle, $[\varphi > \psi] \vee [\varphi > \neg\psi]$, is much more easy to relate to causal notions than Stalnaker's, since it is possible to say (with only a little hedging) that such a conditional is true in case there is a connection of determination of some sort between the antecedent and the consequent. But it has seemed less easy to reconcile the theory of such a conditional with simple tensed examples, like the case of Jim and Jack, invented much earlier by Downing [1959].

Jim and Jack quarrelled yesterday; Jack is unforgiving, and Jim is proud. The example is this.

- (25) If Jim were to ask Jack for help today,
Jack would help him.

Most authors feel that (25) could be taken in two ways. It could be taken to be false, because they quarrelled and Jack is so unforgiving. It could be taken to be true, because Jim is so proud that if he were to ask for help they would not have quarrelled yesterday. But the preferred understanding of (25) seems to be the first of these; and this is only one way in which a systematic preference for alternatives that involve only small changes in the past⁴¹ seems to affect our habits of evaluating such conditionals. An examination of such preferences and their influence on the sort of similarity that is involved in interpreting conditions can be found in [Lewis, 1979a]. Further information can be found in [Bennett, 1982; Thomason, 1982].

This, of course, relates conditionals to time in a philosophical way. But Lewis' informal way of attacking the problem assumes that the logic has

³⁹Judging from some unpublished manuscripts that I have recently received, it is likely to receive more before long. But these are working drafts, which I should not discuss here.

⁴⁰See [Sosa, 1975] for a collection juxtaposing some recent papers on causality with ones on conditionals. (Many of the papers seek to establish links between the two.) Also see [Downing, 1959; Jackson, 1977].

⁴¹Though I put it this way D. Lewis (who assumes determinism for the sake of argument in [Lewis, 1979a], so that changes in the future must be accompanied by changes in the past) has to resort to a hierarchy of maxims to achieve a like effect. (In particular, Maxim 1, [Lewis, 1979a, p. 472], enjoins us to avoid wholesale violations of law, and Maxim 2 tells us to seek widespread perfect match of particular fact; together, these have much the same effect as the principle I stated in the text.

to come to an end, and in particular that there are no new validities to be discovered by placing conditionals in a temporal setting.⁴² In view of the lessons we have learned about the combination of tense with other modalities, this may be a methodological oversight; it runs the risk of not getting the most out of the possible worlds semantics that is to be put to philosophical use.⁴³

Thomason and Gupta [1981] and Van Fraassen [1981] are two recent studies that pursue this model theoretic route: the former uses treelike frames and the latter a version of the $T \times W$ approach. The two treatments are very similar in essentials, though a decision about how to secure the validity of (27) leads to much technical complexity in [Thomason and Gupta, 1981]. Since Van Fraassen's exposition is so clear, and I fear I would be repeating myself if I attempted a detailed account of [Van Fraassen, 1981], I will be very brief.

Both papers endorse certain validities involving a mixture of tense, historical necessity, and the conditional. The following examples are representative.

$$(26) \quad [\Diamond\varphi \wedge [\varphi > \psi]] \rightarrow \Diamond\psi$$

$$(27) \quad [\Box\neg\varphi \wedge \Box[\varphi > \psi]] \rightarrow [\varphi \rightarrow \Box[\varphi \rightarrow \psi]]$$

The first of these represents one way in which selection principles for $>$ can be formulated in terms of alternative histories; the validity of (26) corresponds to a preference at $\langle t, w \rangle$ for alternatives that are possible futures for $\langle t, w \rangle$.

Example (27) called the Edelberg inference after Walter Edelberg, who first noticed it, represents a principle of 'conditional transmission of settledness' that can be made quite plausible; see the discussion in Thomason and Gupta [1981, pp. 306–307]. Formally, its validity depends on there being no 'unattached' counterfactual futures—ones that are not picked out as counterfactual alternatives on condition φ with respect to actual futures. At least, this is the way that Van Fraassen secures the validity of (28); Thomason and Gupta do it in a circuitous way, at the cost of making their theory much more difficult to explain.

If there advantages to offset this cost, they have to do with causality. The key notion introduced in [Thomason and Gupta, 1981], which is not

⁴²I mean that in [Lewis, 1979a], Lewis phrases the discussion in terms of 'similarity'. This is the intuitive notion used to explicate the technical gadgets (assignments of sets of possible worlds to each world) that yield Lewis' theory in [Lewis, 1973] of the satisfaction conditions for the conditional. In [Lewis, 1979a], he leaves things at this informal level, and doesn't try to build temporal conditional frames which can be used to define satisfaction for an extended language.

⁴³If Lewis' adoption of a determinist position in [Lewis, 1979a] is not for the sake of argument, there may be a philosophical issue at work here. A philosophical determinist would be much more likely to follow an approach like Lewis' than to base conditional logic on the logic of historical necessity.

needed by Van Fraassen,⁴⁴ is that of a *future choice function*: a function $\mathcal{F}$ that for each moment t in a treelike frame chooses a branch in B_t . future choice functions must choose coherently, so that if $t' \in \mathcal{F}_t$ then $\mathcal{F}_t = \mathcal{F}_{t'}$. By considering restricted sets of choice functions, Thomason and Gupta are able to introduce a modal notion that, they claim, may help to explicate causal independence. If this claim could be made good it would be worth the added complexity, since so far the techniques of possible worlds semantics have not been of much direct help in clarifying the philosophical debate about causality. But in [Thomason and Gupta, 1981], the idea is not developed enough to see very clearly what the prospects of success are.

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EDITORIAL NOTE

The present chapter is reproduced from the first edition of the Handbook. A continuation chapter will appear in a later volume of the present, second edition. The logic of historical necessity is technically a special combination of modality and temporal operators. Combinations with temporal logic, (or *temporalising*) are discussed in chapter 2 of this volume. Branching temporal logics with modalities in the spirit of the logics of this chapter have been very successfully introduced in theoretical computer science. These are the CTL (computation tree logic) family of logics. For a survey, see the chapter by Colin Stirling in Volume 2 of the *Handbook of Logic in Computer Science*, S. Abramsky, D. Gabbay and T. Maibaum, editors, pp. 478–551. Oxford University Press, 1992.

The following two sources are also of interest:

1. E. Clarke, Jr., O. Grumberg and D. Peled. *Model Checking*, MIT Press, 2000.
2. E. Clarke and B.-H. Schlingloff. Model checking. In *Handbook of Automated Reasoning*, A. Robinson and a. Voronkov, eds. pp. 1635 and 1790. Elsevier and MIT Press, 2001.

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⁴⁴And which also would not be required on a strict theory of the conditional, such as D. Lewis'. It is important to note in the present context that Thomason and Gupta, as well as Van Fraassen, are working with Stalnaker's theory.

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PHILOSOPHICAL PERSPECTIVES ON QUANTIFICATION IN TENSE AND MODAL LOGIC

INTRODUCTION

The trouble with modal logic, according to its critics, is quantification into modal contexts—i.e. *de re* modality. For on the basis of such quantification, it is claimed, essentialism ensues, and perhaps a bloated universe of *possibilia* as well. The essentialism is avoidable, these critics will agree, but only by turning to a Platonic realm of individual concepts whose existence is no less dubious or problematic than mere *possibilia*. Moreover, basing one's semantics on individual concepts, it is claimed, would in effect render all identity statements containing only proper names either necessarily true or necessarily false—i.e. there would then be no contingent identity statements containing only proper names.

None of these claims is true quite as it stands, however; and in what follows we shall attempt to separate the chaff from the grain by examining the semantics of (first-order) quantified modal logic in the context of different philosophical theories. Beginning with the primary semantics of logical necessity and the philosophical context of logical atomism, for example, we will see that essentialism not only does not ensue but is actually rejected in that context by the validation of the modal thesis of anti-essentialism, and that in consequence all *de re* modalities are reducible to *de dicto* modalities.

Opposed to logical atomism, but on a par with it in its referential interpretation of quantifiers and proper names, is Kripke's semantics for what he properly calls metaphysical necessity. Unlike the primary semantics of logical necessity, in other words, Kripke's semantics for metaphysical necessity is in direct conflict with some of the basic assumptions of logical atomism; and in the form which that conflict takes, which we shall refer to here as the form of a *secondary* semantics for necessity, Kripke's semantics amounts to the initial step toward a proper formulation of Aristotelian essentialism. (A secondary semantics for necessity stands to the primary semantics in essentially the same way that non-standard models for second-order logic stand to standard models.) The problem with this initial step toward Aristotelian essentialism, however, is the problem of all secondary semantics; viz. that of its objective, as opposed to its merely formal, significance—a problem which applies all the more so to Kripke's deepening of his formal semantics by the introduction of an accessibility relation between possible worlds. This, in fact, is the real problem of essentialism.

There are no individual concepts, it will be noted in what follows, in either logical atomism or Kripke's implicit philosophical semantics, and yet in both contexts proper names are rigid designators; that is, in both there can be no contingent identity statements containing only proper names. One need not, accordingly, turn to a Platonic realm of individual concepts in order to achieve this result. Indeed, quite the opposite is the case. That is, it has in fact been for the defence of contingent identity, and not its rejection, that philosophical logicians have turned to a Platonic realm of individual concepts, since, on this view, it is only through the mere coincidence of the denotations of the individual concepts expressed by proper names that an identity statement containing those names can be contingent. Moreover, unless such a Platonic realm is taken as the intensional counterpart of logical atomism (a marriage of dubious coherence), it will not validate the modal thesis of anti-essentialism. That is, one can in fact base a Platonic or logical essentialism—which is not the same thing at all as Aristotelian essentialism—upon such a realm. However, under suitable assumptions, essentialism can also be avoided in such a realm; or rather it can in the weaker sense in which, given these assumptions, all *de re* modalities are reducible to *de dicto* modalities.

Besides the Platonic view of intensionality, on the other hand, there is also a socio-biologically based conceptualist view according to which concepts are not independently existing Platonic forms but cognitive capacities or related structures of the human mind whose realisation in thought is what informs a mental act with a predicable or referential nature. This view, it will be seen, provides an account in which there can be contingent identity statements, but not such as to depend on the coincidence of individual concepts in the platonic sense. Such a conceptualist view will also provide a philosophical foundation for quantified tense logic and paradigmatic analyses thereby of metaphysical modalities in terms of time and causation. The problem of the objective significance of the secondary semantics for the analysed modalities, in other words, is completely resolved on the basis of the nature of time, local or cosmic. The related problem of a possible ontological commitment to *possibilia*, moreover, is in that case only the problem of how conceptualism can account for direct references to past or future objects.

1 THE PRIMARY SEMANTICS OF LOGICAL NECESSITY

We begin by describing what we take to be the primary semantics of logical necessity. Our terminology will proceed as a natural extension of the syntax and semantics of standard first-order logic with identity. Initially, we shall assume that the only singular terms are individual variables. As primitive *logical constants* we take $\rightarrow$, $\neg$, $\forall$, $=$, and $\Box$ for the material conditional sign, the negation sign, the universal quantifier, the identity sign and the

necessity sign, respectively. (The conjunction, disjunction, biconditional, existential quantifier and possibility signs— $\wedge, \vee, \leftrightarrow, \exists$ and $\Diamond$, respectively—are understood to be defined in the usual way as metalinguistic abbreviatory devices.) The only non-logical or *descriptive constants* at this point are predicates of arbitrary (finite) degree. We call a set of such predicates a *language* and understand the well-formed formulas (*wffs*) of a language to be defined in the usual way.

A *model* $\mathfrak{A}$ indexed by a language L , or for brevity, an *L-model*, is a structure of the form $\langle D, R \rangle$, where D , the *universe* of the model, is a non-empty set and R is a function with L as domain and such that for each positive integer n and each n -place predicate F^n in L , $R(F^n) \subseteq D^n$, i.e. $R(F^n)$ is a set of n -tuples of members of D . An *assignment in D* is a function A with the set of individual variables as domain and such that $A(x) \in D$, for each variable x . Where $d \in D$, we understand $A(d/x)$ to be that assignment in D which is exactly like A except for its assigning d to x . The *satisfaction* of a wff φ of L *in $\mathfrak{A}$ by an assignment A in D* , in symbols $\mathfrak{A}, A \models \varphi$, is recursively defined as follows:

1. $\mathfrak{A}, A \models (x = y)$ iff $A(x) = A(y)$;
2. $\mathfrak{A}, A \models P^n(x_1, \dots, x_n)$ iff $\langle A(x_1), \dots, A(x_n) \rangle \in R(P^n)$;
3. $\mathfrak{A}, A \models \neg\varphi$ iff $\mathfrak{A}, A \not\models \varphi$;
4. $\mathfrak{A}, A \models (\varphi \rightarrow \psi)$ iff either $\mathfrak{A}, A \not\models \varphi$ or $\mathfrak{A}, A \models \psi$;
5. $\mathfrak{A}, A \models \forall x\varphi$ iff for all $d \in D$, $\mathfrak{A}, A(d/x) \models \varphi$; and
6. $\mathfrak{A}, A \models \Box\varphi$ iff for all R' , if $\langle D, R' \rangle$ is an L -model, then $\langle D, R' \rangle, A \models \varphi$.

The *truth* of a wff in a model (indexed by a language suitable to that wff) is as usual the satisfaction of the wff by every assignment in the universe of the model. *Logical truth* is then truth in every model (indexed by any appropriate language). One or another version of this primary semantics for logical necessity, it should be noted, occurs in [Carnap, 1946]; [Kanger, 1957]; [Beth, 1960] and [Montague, 1960].

2 LOGICAL ATOMISM AND QUANTIFIED MODAL LOGIC

These definitions, as already indicated, are extensions of essentially the same semantical concepts as defined for the modal free wffs of standard first-order predicate logic with identity. The clause for the necessity operator has a particularly natural motivation within the framework of logical atomism. In such a framework, a model $\langle D, R \rangle$ for a language L represents a *possible world* of a *logical space* based upon (1) D as the universe of objects of that space and (2) L as the predicates characterising the *atomic states of affairs*

of that space. So based, in other words, a logical space consists of the totality of atomic states of affairs all the constituents of which are in D and the characterising predicates of which are in L . A possible world of such a logical space then amounts in effect to a partitioning of the atomic states of affairs of that space into two cells: those that obtain in the world in question and those that do not.

Every model, it is clear, determines both a unique logical space (since it specifies both a domain and a language) and a possible world of that space. In this regard, the clause for the necessity operator in the above definition of satisfaction is the natural extension of the standard definition and interprets that operator as ranging over *all* the possible worlds (models) of the logical space to which the given one belongs.

Now it may be objected that logical atomism is an inappropriate framework upon which to base a system of quantified modal logic; for if any framework is a paradigm of anti-essentialism, it is logical atomism. The objection is void, however, since in fact the above semantics provides the clearest validation of the modal thesis of anti-essentialism. Quantified modal logic, in other words, does not in itself commit one to any non-trivial form of essentialism (cf. [Parsons, 1969]).

The general idea of the modal thesis of anti-essentialism is that if a predicate expression or open wff φ *can be true of* some individuals in a given universe (satisfying a given identity-difference condition with respect to the variables free in φ), then φ *can be true of* any individuals in that universe (satisfying the same identity-difference conditions). In other words, no conditions are essential to some individuals that are not essential to all, which is as it should be if necessity means logical necessity.

The restriction to identity-difference conditions mentioned (parenthetically) above can be dropped, it should be noted, if nested quantifiers are interpreted exclusively and not (as we have done) inclusively where, e.g. it is allowed that the value of y in $\forall x \exists y \varphi(x, y)$ can be the same as the value of x . (Cf. [Hintikka, 1956] for a development of the exclusive interpretation.) Indeed, as Hintikka has shown, when nested quantifiers are interpreted exclusively, identity and difference wffs are superfluous—which is especially apropos of logical atomism where an identity wff does not represent an atomic state of affairs. (Cf. Wittgenstein's *Tractatus Logico-Philosophicus* 5.532–5.53 and [Cocchiarella, 1975a, Section V].)

Retaining the inclusive interpretation and identity as primitive, however, an *identity-difference condition* for distinct individual variables $x_1, \dots, x_n$ is a conjunction of one each but not both of the wffs $(x_i = x_j)$ or $(x_i \neq x_j)$, for all i, j such that $1 \leq i < j \leq n$. It is clear of course that such a conjunction specifies a complete identity-difference condition for the variables $x_1, \dots, x_n$. Since there are only a finite number of non-equivalent such conditions for $x_1, \dots, x_n$, moreover, we understand $ID_j(x_1, \dots, x_n)$, relative to an assumed ordering of such non-equivalent conjunctions, to be the j th

conjunction in the ordering. *The modal thesis of anti-essentialism* may now be stated as the thesis that every wff of the form

$$\begin{aligned} & \exists x_1 \dots \exists x_n (\text{ID}_j(x_1, \dots, x_n) \wedge \Diamond \varphi) \\ & \rightarrow \forall x_1 \dots \forall x_n (\text{ID}_j(x_1, \dots, x_n) \rightarrow \Diamond \varphi) \end{aligned}$$

is to be logically true, where $x_1, \dots, x_n$ are all the distinct individual variables occurring free in φ . (Where $n = 0$, the above wff is understood to be just $(\Diamond \varphi \rightarrow \Diamond \varphi)$; and where $n = 1$, it is understood to be just $\exists x \Diamond \varphi \rightarrow \forall x \Diamond \varphi$.) The validation of the thesis in our present semantics is easily seen to be a consequence of the following lemma (whose proof is by a simple induction on the wffs of L).

LEMMA If L is a language, $\mathfrak{A}, \mathfrak{B}$ are L-models, and h is an isomorphism of $\mathfrak{A}$ with $\mathfrak{B}$, then for all wffs φ of L and all assignments A in the universe of $\mathfrak{A}$, $\mathfrak{A}, A \models \varphi$ iff $\mathfrak{B}, A/h \models \varphi$.

One of the nice consequences of the modal thesis of anti-essentialism in the present semantics, it should be noted, is the reduction of all *de re* wffs to *de dicto* wffs. (A *de re* wff is one in which some individual variable has a free occurrence in a subwff of the form $\Box \psi$. A *de dicto* wff is a wff that is not *de re*.) Naturally, such a consequence is a further sign that all is well with our association of the present semantics with logical atomism.

THEOREM (De Re Elimination Theorem) For each *de re* wff φ , there is a *de dicto* wff ψ such that $(\varphi \leftrightarrow \psi)$ is logically true.¹

These niceties aside, however, another result of the present semantics is its essential incompleteness with respect to any language containing at least one relational predicate. (It is not only complete but even decidable when restricted to monadic wffs—of which more anon.) The incompleteness is easily seen to follow from the following lemma and the well-known fact that the modal free non-logical truths of a language containing at least one relational predicate is not recursively enumerable (cf. [Cocchiarella, 1975b]). (It is also for the statement of the infinity condition of this lemma that a relational predicate is needed.)

LEMMA If ψ is a sentence which is satisfiable, but only in an infinite model, and φ is a modal and identity-free sentence, then $(\psi \rightarrow \neg \Box \varphi)$ is logically true iff φ is not logically true.

¹A proof of this theorem can be found in [McKay, 1975]. Briefly, where $x_1, \dots, x_n$ are all the distinct individual variables occurring free in φ and $ID_1(x_1, \dots, x_n), \dots, ID_k(x_1, \dots, x_n)$ are all the non-equivalent identity-difference conditions for $x_1, \dots, x_n$, then the equivalence in question can be shown if ψ is obtained from φ by replacing each subwff $\Box \chi$ of φ by:

$$\begin{aligned} & [ID_1(x_1, \dots, x_n) \wedge \Box \forall x_1 \dots \forall x_n (ID_1(x_1, \dots, x_n) \rightarrow \chi)] \vee \dots \\ & \vee [ID_k(x_1, \dots, x_n) \wedge \Box \forall x_1 \dots \forall x_n (ID_k(x_1, \dots, x_n) \rightarrow \chi)]. \end{aligned}$$

THEOREM If L is a language containing at least one relational predicate, then the set of wffs of L that are logically true is not recursively enumerable.

This last result does not affect the association we have made of the primary semantics with logical atomism. Indeed, given the Löwenheim–Skolem theorem, what this lemma shows is that there is a complete concurrence between logical necessity as an internal condition of modal free propositions (or of their corresponding states of affairs) and logical truth as a semantical condition of the modal free sentences expressing those propositions (or representing their corresponding states of affairs). And that of course is as it should be if the operator for logical necessity is to have only formal and no material content.

Finally, it should be noted that the above incompleteness theorem explains why Carnap was not able to prove the completeness of the system of quantified modal logic formulated in [Carnap, 1946]. For on the assumption that the number of objects in the universe is denumerably infinite, Carnap's state description semantics is essentially that of the primary semantics restricted to denumerably infinite models; and, of course, precisely because the models are denumerably infinite, the above incompleteness theorem applies to Carnap's formulation as well. Thus, the reason why Carnap was unable to carry through his proof of completeness is finally answered.

3 THE SECONDARY SEMANTICS OF METAPHYSICAL NECESSITY

Like the situation in standard second-order logic, the incompleteness of the primary semantics can be avoided by allowing the quantificational interpretation of necessity in the metalanguage to refer not to *all* the possible worlds (models) of a given logical space but only to those in a given non-empty set of such worlds. Of course, since a model may belong to many such sets, the relativisation to the one in question must be included as part of the definition of satisfaction.

Accordingly, where L is a language and D is a non-empty set, we understand a *model structure* based on D and L to be a pair $\langle \mathfrak{A}, K \rangle$, where K is a set of L -models all having D as their universe and $\mathfrak{A} \in K$. The *satisfaction* of a wff φ of L in such a model structure by an assignment A in D , in symbols $\langle \mathfrak{A}, K \rangle, A \models \varphi$, is recursively defined exactly as in Section 1, except for clause (6) which is defined as follows:

6. $\langle \mathfrak{A}, K \rangle, A \models \Box\varphi$ iff for all $\mathfrak{B} \in K$, $\langle \mathfrak{B}, K \rangle, A \models \varphi$.

Instead of logical truth, a wff is understood to be *universally valid* if it is satisfied by every assignment in every model structure based on a language to which the wff belongs. Where **QS5** is standard first-order logic with

identity supplemented with the axioms of **S5** propositional modal logic, a completeness theorem for the secondary semantics of logical necessity was proved by Kripke in [1959].

THEOREM (Completeness Theorem). A set Γ of wffs is consistent in **QS5** iff all the members of Γ are simultaneously satisfiable in a model structure; and (therefore) a wff φ is a theorem of **QS5** iff φ is universally valid.

The secondary semantics, despite the above completeness theorem, has too high a price to pay as far as logical atomism is concerned. In particular, unlike the situation in the primary semantics, the secondary semantics does not validate the modal thesis of anti-essentialism—i.e. it is false that every instance of the thesis is universally valid. This is so of course because necessity no longer represents an invariance through all the possible worlds of a given logical space but only through those in arbitrary non-empty sets of such worlds; that is, necessity is now allowed to represent an internal condition of propositions (or of their corresponding states of affairs) which has material and not merely formal content—for what is invariant through all the members of such a non-empty set need not be invariant through all the possible worlds (models) of the logical space to which those in the set belong.

One example of how such material content affects the implicit metaphysical background can be found in monadic modal predicate logic. It is well-known, for example, that modal free monadic predicate logic is decidable and that no modal free monadic wff can be true in an infinite model unless it is true in a finite model as well. Consequently, any substitution instance of a modal free monadic wff for a relational predicate in an infinity axiom is not only false but logically false. It follows, accordingly, that there can be no modal free analysis or reduction otherwise of all relational predicates or open wffs in terms only of monadic predicates, i.e. in terms only of modal free monadic wffs.

Now it turns out that the same result also holds in the primary semantics for quantified modal logic. That is, in the primary semantics, modal monadic predicate logic is also decidable and no monadic wff, modal free or otherwise, can be true in an infinite model unless it is also true in a finite model (cf. [Cocchiarella, 1975b]). Consequently, there can also be no modal analysis or reduction otherwise of all relational predicates or open wffs in terms only of monadic wffs, modal free or otherwise.

With respect to the secondary semantics, however, the situation is quite different. In particular, as Kripke has shown in [1962], modal monadic predicate logic, as interpreted in the secondary semantics, is not decidable. Moreover, on the basis of that semantics a modal analysis of relational predicates in terms of monadic predicates can in general be given. E.g.,

substituting $\Diamond(Fx \wedge Gy)$ for the binary predicate R in the infinity axiom

$$\begin{aligned} & \forall x \neg R(x, x) \wedge \forall x \exists y R(x, y) \wedge \forall x \forall y \forall z [R(x, y) \\ & \quad \wedge R(y, z) \rightarrow R(x, z)] \end{aligned}$$

results in a modal monadic sentence which is true in some model structure based on an infinite universe and false in all model structures based on a finite domain. Somehow, in other words, relational content has been incorporated in the semantics for necessity, and thereby of possibility as well. In this respect, the secondary semantics is not the semantics of a merely formal or logical necessity but of a necessity having additional content as well.

Kripke himself, it should be noted, speaks of the necessity of his semantics not as a formal or logical necessity but as a metaphysical necessity (cf. [Kripke, 1971, p. 150]). Indeed, it is precisely because he is concerned with a metaphysical or material necessity and not a logical necessity that not every necessary proposition needs to be *a priori*, nor every *a posteriori* proposition contingent (*ibid.*). Needless to say, however, but that the latter result should obtain does not of itself amount to a refutation, as it is often taken, of the claim of logical atomism that every logically necessary proposition is *a priori* and that every *a posteriori* proposition is logically contingent. We are simply in two different metaphysical frameworks, each with its own notion of necessity and thereby of contingency as well.

4 PROPER NAMES AS RIGID DESIGNATORS

Ordinary proper names in the framework of logical atomism are not what Bertrand Russell called ‘logically proper names’, because the things they name, if they name anything at all, are not the simple objects that are the constituents of atomic states of affairs. However, whereas the names of ordinary language have a sense (*Sinn*) insofar as they are introduced into discourse with identity criteria (usually provided by a sortal common noun with which they are associated—cf. [Geach, 1962, p. 43 f]), the logically proper names of logical atomism have no sense other than what they designate. In other words, in logical atomism, ‘a name means (*bedeutet*) an object. The object is its meaning (*Bedeutung*)’ (*Tractatus* 3.203). Different identity criteria have no bearing on the simple objects of logical atomism, and (pseudo) identity propositions, strictly speaking, have no sense (*Sinn*)—i.e. they do not represent an atomic state of affairs.

Semantically, what this comes to is that logically proper names, or *individual constants*, are rigid designators; that is, their introduction into formal languages requires that the wff

$$\exists x \Box (a = x)$$

be logically true in the primary semantics for each individual constant a . Carnap, in his formulation of the primary semantics, also required that $(a \neq b)$ be logically true for distinct individual constants; but that was because his semantics was given in terms of state descriptions where redundant proper names have a complicating effect. Carnap's additional assumption that there is an individual constant for every object in the universe is, of course, also an assumption demanded by his use of state descriptions and is not required by our present model-theoretic approach. (It is noteworthy, however, that the assumption amounted in effect to perhaps the first substitution interpretation of quantifiers, and that in fact it was Carnap who first observed that a strong completeness theorem even for modal free wffs could not be established for an infinite domain on the basis of such an interpretation. Cf. [Carnap, 1938, p. 165].)

Kripke also claims that proper names are rigid designators, but his proper names are those of ordinary language and, as already noted, his necessity is a metaphysical and not a logical necessity. Nevertheless, in agreement with logical atomism the function of a proper name, according to Kripke, is simply to refer, and not to describe the object named [Kripke, 1971, p. 140]; and this applies even when we fix the reference of a proper name by means of a definite description—for the relation between a proper name and a description used to fix the reference of the name is not that of synonymy [Kripke, 1971, p. 156f]. To the objection that we need a criterion of identity across possible worlds before we can determine whether a name is rigid or not, Kripke notes that we should distinguish *how we would speak in* a counterfactual situation from *how we do speak of* a counterfactual situation [Kripke, 1971, p. 159]. That is, the problem of cross-world identity, according to Kripke, arises only through confusing the one way of speaking with the other and that it is otherwise only a pseudo-problem.

5 NON-CONTINGENT IDENTITY AND THE CARNAP–BARCAN FORMULA

As rigid designators, proper names cannot be the only singular terms occurring in contingent identity statements. That is, a contingent identity statement must contain at least one definite description whose descriptive content is what accounts for the possibility of different designata and thereby of the contingency of the statement in question. However, in general, as noted by [Smullyan, 1948], there is no problem about contingent identity in quantified modal logic if one of the singular terms involved is a definite description; or rather there is no problem so long as one is careful to observe the proper scope distinctions. On the other hand, where scope distinctions are not assumed to have a bearing on the occurrence of a proper name, the problem of contingent identity statements involving only proper names is trivially resolved by their construal as rigid designators. That is, where a

and b are proper names or individual constants, the sentences

$$(a = b) \rightarrow \Box(a = b), \quad (a \neq b) \rightarrow \Box(a \neq b)$$

are to be logically true in the primary semantics and universally valid in the secondary. In other words, whether in the context of logical atomism or Kripke's metaphysical necessity, there are only non-contingent identity statements involving only proper names.

Now it is noteworthy that the incorporation of identity-difference conditions in the modal thesis of anti-essentialism disassociates these conditions from the question of essentialism. This is certainly as it should be in logical atomism, since in that framework, as F. P. Ramsey was the first to note, 'numerical identity and differences are necessary relations' [Ramsey, 1960, p. 155]. In other words, even aside from the use of logically proper names, the fact that there can be no contingent identities or non-identities in logical atomism is reflected in the logical truth of both of the wffs

$$\forall x \forall y (x = y \rightarrow \Box x = y), \quad \forall x \forall y (x \neq y \rightarrow \Box x \neq y)$$

in the primary semantics. But then even in the framework of Kripke's metaphysical necessity (where quantifiers also refer *directly* to objects), an object cannot but be the object that it is, nor can one object be identical with another—a metaphysical fact which is reflected in the above wffs being universally valid as well.

Another observation made by Ramsey in his adoption of the framework of logical atomism was that the number of objects in the world is part of its logical scaffolding [Ramsey, 1960]. That is, for each positive integer n , it is either necessary or impossible that there are exactly n individuals in the world; and if the number of objects is infinite, then, for each positive integer n , it is necessary that there are at least n objects in the world (cf. [Cocchiarella, 1975a, Section 5]. This is so in logical atomism because every possible world consists of the same totality of objects that are the constituents of the atomic states of affairs constituting the actual world. In logical atomism, in other words, an object's existence is not itself an atomic state of affairs but consists in that object's being a constituent of atomic states of affairs.

One important consequence of the fact that every possible world (of a given logical space) consists of the same totality of objects is the logical truth in the primary semantics of the well-known Barcan formula (and its converse):

$$\forall x \Box \varphi \leftrightarrow \Box \forall x \varphi.$$

Carnap, it should be noted, was the first to argue for the logical truth of this principle (in [Carnap, 1946, Section 10] and [1947, Section 40]) which he validated in terms of the substitution interpretation of quantifiers in

his state description semantics. The validation does not depend, of course, on the number of objects being denumerably infinite, though, as already noted, Carnap did impose that condition on his state descriptions. But then—even though Carnap himself did not give this argument—given the non-contingency of identity, the logical truth of the Carnap–Barcan formula, and the assumption for each positive integer n that it is not necessary that there are just n objects in the world, it follows that the number of objects in the world must be infinite. (For if everything is one of a finite number n of objects, then, by the non-contingency of identity, everything is necessarily one of n objects, and therefore by the Carnap–Barcan formula, necessarily everything is one of n objects; i.e. contrary to the assumption, it is necessary after all that there are just n objects in the world.)

As the above remarks indicate, the validation of the Carnap–Barcan formula in the framework of logical atomism is unproblematic; and therefore its logical truth in the primary semantics is as it should be. However, besides being logically true in the primary semantics the principle is also universally valid in the secondary semantics; and it is not clear that this is as it should be for Kripke’s metaphysical necessity. Indeed, Kripke’s later modified semantics for quantified modal logic in [Kripke, 1963] suggests he thinks otherwise, since there the Carnap–Barcan formula is no longer validated. Nevertheless, as indicated above, even with the rejection of the Carnap–Barcan formula, it is clear that Kripke intends his metaphysical context to be such as to support the validation of the non-contingency of identity.

6 EXISTENCE IN THE PRIMARY AND SECONDARY SEMANTICS

In rejecting the Carnap–Barcan formula, one need not completely reject the assumption upon which it is based, *viz.* that every possible world (of a given logical space) consists of the same totality of objects. All one need do is take this totality not as the set of objects existing in each world but as the sum of objects that exist in some world or other (of the same logical space), i.e. as the totality of possible objects (of that logical space). Quantification with respect to a world, however, is always to be restricted to the objects existing in that world—though free variables may, as it were, range over the possible objects, thereby allowing a single interpretation of both *de re* and *de dicto* wffs. The resulting quantificational logic is of course free of the presupposition that singular terms (individual variables and constants) always designate an existing object and is for this reason called *free logic* (cf. [Hintikka, 1969]).

Thus, where L is a language and D is a non-empty set, then $\langle \langle \mathfrak{A}, X \rangle, K \rangle$ is a *free model structure* based on D and L iff (1) $\langle \mathfrak{A}, X \rangle \in K$, (2) K is a set every member of which is a pair $\langle \mathfrak{B}, Y \rangle$ where $\mathfrak{B}$ is an L -model having

D as its universe and $Y \subseteq D$ and (3) $D = \cup\{Y : \text{for some L-model } \mathfrak{B}, \langle \mathfrak{B}, Y \rangle \in K\}$. Possible worlds are now represented by the pairs $\langle \mathfrak{B}, Y \rangle$, where the (possibly empty) set Y consists of just the objects existing in the world in question; and of course the pair $\langle \mathfrak{A}, X \rangle$ is understood to represent the actual world. Where A is an assignment in D , the *satisfaction by* A of a wff φ of L in $\langle \langle \mathfrak{A}, X \rangle, K \rangle$ is defined as in the secondary semantics, except for clause (5) which is now as follows:

5. $\langle \langle \mathfrak{A}, X \rangle, K \rangle, A \models \forall x \varphi$ iff for all $d \in X$, $\langle \langle \mathfrak{A}, X \rangle, K \rangle, A(d/x) \models \varphi$.

Now if K is the set of *all* pairs $\langle \mathfrak{B}, Y \rangle$, where $\mathfrak{B}$ is an L-model having D as its universe and $Y \subseteq D$, then $\langle \langle \mathfrak{A}, X \rangle, K \rangle$ is a *full* free model structure. Of course, whereas validity with respect to all free model structures (based on an appropriate language) is the free logic counterpart of the secondary semantics, validity with respect to all full free model structures is the free logic version of the primary semantics. Moreover, because of the restricted interpretation quantifiers are now given, neither the Carnap–Barcan formula nor its converse is valid in either sense, i.e. neither is valid in either the primary or secondary semantics based on free logic.

If a formal language L contains proper names or individual constants, then their construal as rigid designators requires that a free model structure $\langle \langle \mathfrak{A}, X \rangle, K \rangle$ based on L be such that for all $\langle \mathfrak{B}, Y \rangle \in K$ and all individual constants a in L , the designation of a in $\mathfrak{B}$ is the same as the designation of a in $\mathfrak{A}$, i.e. in the actual world. Note that while it is assumed that every individual constant designates a possible object, i.e. possibly designates an existing object, it need not be assumed that it designates an existing object, i.e. an object existing in the actual world. In that case, the rigidity of such a designator is not given by the validity of $\exists x \Box(a = x)$ but by the validity of $\Diamond \exists x \Box(a = x)$ instead. Existence of course is analysable as follows:

$$E!(a) = df \exists x(a = x).$$

Note that since possible worlds are now differentiated from one another by the objects existing in them, the concept of existence, despite its analysis in logical terms, must be construed here as having material and not merely formal content. In logical atomism, however, that would mean that the existence or non-existence of an object is itself an atomic state of affairs after all, since now even merely possible objects are constituents of atomic states of affairs. To exclude the later situation, i.e. to restrict the constituents of atomic states of affairs to those that exist in the world in question, would mean that merely possible worlds are not after all merely alternative combinations of the same atomic states of affairs that constitute the actual world; that is, it would involve rejecting one of the basic features of logical atomism, and indeed one upon which the coherence of the framework depends. In this regard, it should be noted, while it is one thing to reject logical

atomism (as probably most of us do) as other than a paradigm of logical analysis, it is quite another to accept some of its basic features (such as the interpretation of necessity as referring to *all* the possible worlds of a given logical space) while rejecting others (such as the constitutive nature of a possible world); for in that case, even if it is set-theoretically consistent, it is no longer clear that one is dealing with a philosophically coherent framework.

That existence should have material content in the secondary semantics, on the other hand is no doubt as it should be, since as already noted, necessity is itself supposed to have such content in that semantics. The difficulty here, however, is that necessity can have such content in the secondary semantics only in a free model structure that is *not* full; for with respect to the full free model structure, the modal thesis of anti-essentialism (with quantifiers now interpreted as respecting existing objects only) can again be validated, just as it was in the original primary semantics. (A full free model structure, incidentally, is essentially what Parsons in [1969] calls a maximal model structure.) The key lemma that led to its validation before continues to hold, in other words, only now for free model structures $\langle \langle \mathfrak{A}, X \rangle, K \rangle$ and $\langle \langle \mathfrak{B}, Y \rangle, K \rangle$ instead of the models $\mathfrak{A}$ and $\mathfrak{B}$, and for an isomorphism h between $\mathfrak{A}$ and $\mathfrak{B}$ such that $Y = h''(X)$. Needless to say, moreover, but the incompleteness theorem of the primary semantics for logical truth also carries over to universal validity with respect to all full free model structures.

No doubt one can attempt to avoid this difficulty by simply excluding full free model structures; but that in itself would hardly constitute a satisfactory account of the metaphysical content of necessity (and now of existence as well). For there remains the problem of explaining how arbitrary non-empty subsets of the set of possible worlds in a free model structure can themselves be the referential basis for necessity in other free model structures. Indeed, in general, the problem with the secondary semantics is that it provides no explanation of why *arbitrary* non-empty sets of possible worlds can be the referential basis of necessity. In this regard, the secondary semantics of necessity is quite unlike the secondary semantics of second-order logic where, e.g. general models are subject to the constraints of the compositional laws of a comprehension principle.

7 METAPHYSICAL NECESSITY AND RELATIONAL MODEL STRUCTURES

It is noteworthy that in his later rejection of the Carnap–Barcan formula, Kripke also introduced a further restriction into the quantificational semantics of necessity, viz. that it was to refer not to all the possible worlds in a given model structure but only to those that are possible alternatives to

the world in question. In other words, not only need not all the worlds in a given logical space be in the model structure (the first restriction), but now even the worlds in the model structure need not all be possible alternatives to one another (the second restriction). Clearly, such a restriction within the first restriction only deepens the sense in which the necessity in question is no longer a logical but a material or metaphysical modality.

The virtue of a relational interpretation, as is now well-known, is that it allows for a general semantical approach to a whole variety of modal logics by simply imposing in each case certain structural conditions on the relation of accessibility (or alternative possibility) between possible worlds. Of course, in each such case, the question remains as to the real nature and content of the structural conditions imposed, especially if our concern is with giving necessity a metaphysical or material interpretation as opposed to a merely formal or set-theoretical one. How this content is explained and filled in, needless to say, will no doubt affect how we are to understand modality *de re* and the question of essentialism.

Retaining the semantical approach of the previous section where the restrictions on possible worlds (models with a restricted existence set) are rendered explicit, we shall understand a *relational model structure* based on a universe D and a language L to be a triple $\langle \langle \mathfrak{A}, X \rangle, K, R \rangle$ where (1) $\langle \langle \mathfrak{A}, X \rangle, K \rangle$ is a free model structure and (2) $R \subseteq K \times K$. If A is an assignment in D , then *satisfaction* by A is defined as in Section 6, except for clause (6) which now is as follows:

6. $\langle \langle \mathfrak{A}, X \rangle, K, R \rangle, A \models \Box \varphi$ iff for all $\langle \mathfrak{B}, Y \rangle \in K$, if $\langle \mathfrak{A}, X \rangle R \langle \mathfrak{B}, Y \rangle$, then $\langle \langle \mathfrak{B}, Y \rangle, K, R \rangle, A \models \varphi$.

Needless to say, but if $R = K \times K$ and K is full, then once again we are back to the free logic version of the primary semantics; and, as before, even excluding relational model structures that are full in this extended sense still leaves us with the problem of explaining how otherwise arbitrary non-empty sets of possible worlds of a given logical space, together now with a relation of accessibility between such worlds, can be the basis of a metaphysical modality.

No doubt it can be assumed regarding the implicit metaphysical framework of such a modality that in addition to the objects that exist in a given world there are properties and relations which these objects either do or do not have and which account for the truths that obtain in that world. They do so, of course, by being what predicate expressions stand for as opposed to the objects that are the designata of singular terms. (Nominalism, it might be noted, will not result in a coherent theory of predication in a framework which contains a metaphysical modality—a point nominalists themselves insist on.)

On the other hand, being only what predicates stand for, properties and relations do not themselves exist in a world the way objects do. That

is, unlike the objects that exist in a world and which might not exist in another possible world, the properties and relations that are predicable of objects in one possible world are the same properties and relations that are predicable of objects in any other possible world. In this regard, what is semantically peculiar to a world about a property or relation is not the property or relation itself but only its extension, i.e. the objects that are in fact conditioned by that property or relation in the world in question. Understood in this way, a property or relation may be said to have in itself only a transworld or non-substantial mode of being.

Following Carnap [1955], who was the first to make this sort of proposal, we can represent a property or relation in the sense indicated by a function from possible worlds to extensions of the relevant sort. With respect to the present semantics, however, it should be noted that the extension which a predicate expression has in a given world need not be drawn exclusively from the objects that exist in that world. That is, the properties and relations that are part of the implicit metaphysical framework of the present semantics may apply not only to existing objects but to possible objects as well—even though quantification is only with respect to existing objects. Syntactically, this is reflected in the fact that the rule of substitution:

$$\text{if } \models \varphi, \text{ then } \models \check{S}_{\psi}^{F(x_1, \dots, x_n)} \varphi \mid$$

is validated in the present semantics; and this in turn indicates that any open wff ψ , whether *de re* or *de dicto*, may serve as the definiens of a possible definition for a predicate. That is, it can be shown by means of this rule that such a definiens will satisfy both the criterion of eliminability and the criterion of non-creativity for explicit definitions of a new predicate constant. (Beth's Definability Theorem fails for the logic of this semantics, however, and therefore so does Craig's Interpolation Lemma which implies the Definability Theorem, cf. [Fine, 1979].)

We can, of course, modify the present semantics so that the extension of a predicate is always drawn exclusively from existing objects. That perhaps would make the metaphysics implicit in the semantics more palatable—especially if metaphysical necessity in the end amounts to a physical or natural necessity and the properties and relations implicit in the framework are physical or natural properties and relations rather than properties and relations in the logical or intensional sense. However, in that case we must then also give up the above rule of substitution and restrict the conditions on what constitutes a possible explicit definition of a predicate; e.g. not only would modal wffs in general be excluded as possible definiens but so would the negation of any modal free wff which itself was acceptable. In consequence, not only would many of the predicates of natural language not be representable by predicates of a formal modal language—their associated 'properties' being dispositional or modal—but even their possible analyses in terms of predicates that are acceptable would also be excluded.

8 QUANTIFICATION WITH RESPECT TO INDIVIDUAL CONCEPTS

One way out of the apparent impasse of the preceding semantics is the turn to intensionality, i.e. the turn to an independently existing Platonic realm of intensional existence (and inexistence) and away from the metaphysics of either essentialism (natural kinds, physical properties, etc.) or anti-essentialism (logical atomism). In particular, it is claimed, problems about states of affairs, possible objects and properties and relations (in the material sense) between such objects are all avoidable if we would only turn instead to propositions, individual concepts and properties and relations in the logical sense, i.e. as the intensions of predicate expressions and open wffs in general. Thus, unlike the problem of whether there can be a state of affairs having merely possible objects among its constituents, there is nothing problematic, it is claimed, about the intensional existence of a proposition having among its components intensionally inexistent individual concepts, i.e. individual concepts that fail to denote (*bedeuten*) an existing object.

Intensional entities (*Sinne*), on this approach, do not exist in space and time and are not among the individuals that differentiate one possible world from another. They are rather non-substantial transworld entities which, like properties and relations, may have different extensions (*Bedeutungen*) in different possible worlds. For example, the extension of a proposition in a given world is its truth-value, i.e. truth or falsity (both of which we shall represent here by 1 and 0, respectively), and the extension of an individual concept in that world is the object which it denotes or determines. We may, accordingly, follow Carnap once again and represent different types of intensions in general as functions from possible worlds to extensions of the relevant type. In doing so, however, we shall no longer identify possible worlds with the extensional models of the preceding semantics; that is, except for the objects that exist in a given world, the nature and content of that world will otherwise be left unspecified. Indeed, because intensionality is assumed to be conceptually prior to the functions on possible worlds in terms of which it will herein be represented, the question of whether a merely possible world, or of whether a merely possible object existing in such a world, has an ontological status independent of the realm of intensional existence (or inexistence) is to be left open on this approach (if not closed in favour of an analysis or reduction of such worlds and objects in terms of propositions and the intensional inexistence of individual concepts).

Accordingly, a triple $\langle W, R, E \rangle$ will be said to be a *relational world system* if W is a non-empty set of possible worlds, R is an accessibility relation between the worlds in W , i.e. $R \subseteq W \times W$, and E is a function on W into (possibly empty) sets, though for some $w \in W$ (representing the actual world), $E(w)$ is non-empty. (The sets $E(w)$, for $w \in W$, consist of the objects existing in each of the worlds in W .) Where $D = \cup_{w \in W} E(w)$, an

individual concept in $\langle W, R, E \rangle$, is a function in D^W ; and for each natural number n , P is an n -place *predicate intension* in $\langle W, R, E \rangle$ iff $P \in \{X : X \subseteq D^n\}^W$. (Note: for $n = 0$, we take an n -place predicate intension in $\langle W, R, E \rangle$ to be a *proposition* in $\langle W, R, E \rangle$, and therefore since $D^0 = \{0\}$ and $2 = \{0, 1\}$, P is a proposition in $\langle W, R, E \rangle$ iff $P \in 2^W$. For $n \geq 2$, an n -place predicate intension is also called an n -ary *relation-in-intension*, and for $n = 1$, it is taken as a property in the logical sense.)

Where L is a language, I is said to be an *interpretation* for L based on a relational world system $\langle W, R, E \rangle$ if I is a function on L such that (1) for each individual constant $a \in L$, $I(a)$ is an individual concept in $\langle W, R, E \rangle$; and (2) if F^n is an n -place predicate in L , then $I(F^n)$ is an n -place predicate intension in $\langle W, R, E \rangle$. An *assignment* A in $\langle W, R, E \rangle$ is now a function on the individual variables such that $A(x)$ is an individual concept in $\langle W, R, E \rangle$, for each such variable. The *intension* with respect to I and A of an individual variable or constant b in L , in symbols $\text{Int}(b, I, A)$ is defined to be $(I \cup A)(b)$.

Finally, the *intension with respect to I and A of an arbitrary wff φ of L* is defined recursively as follows:

1. where a, b are individual variables or constants in L , $\text{Int}(a = b, I, A) =$ the $p \in 2^W$ such that for $w \in W$, $P(w) = 1$ iff $\text{Int}(a, I, A)(w) = \text{Int}(b, I, A)(w)$;
2. where $a_1, \dots, a_n$ are individual variables or constants in L and $F^n \in L$, $\text{Int}(F(a_1, \dots, a_n), I, A) =$ the $P \in 2^W$ such that for $w \in W$, $P(w) = 1$ iff $\langle \text{Int}(a_1, I, A)(w), \dots, \text{Int}(a_n, I, A)(w) \rangle \in I(F^n)(w)$;
3. $\text{Int}(\neg\varphi, I, A) =$ the $P \in 2^W$ such that for $w \in W$, $P(w) = 1$ iff $\text{Int}(\varphi, I, A)(w) = 0$;
4. $\text{Int}((\varphi \rightarrow \psi), I, A) =$ the $P \in 2^W$ such that for $w \in W$, $P(w) = 1$ iff either $\text{Int}(\varphi, I, A)(w) = 0$ or $\text{Int}(\psi, I, A)(w) = 1$;
5. $\text{Int}(\forall x\varphi, I, A) =$ the $P \in 2^W$ such that for $w \in W$, $P(w) = 1$ iff for *all* individual concepts f in $\langle W, R, E \rangle$, $\text{Int}(\varphi, I, A(f/x))(w) = 1$; and
6. $\text{Int}(\Box\varphi, I, A) =$ the $P \in 2^W$ such that for $w \in W$, $P(w) = 1$ iff for all $v \in W$, if wRv , then $\text{Int}(\varphi, I, A)(v) = 1$.

Different notions of *intensional validity*, needless to say, can now be defined as in the earlier semantics depending on the different structural properties that the relation of accessibility might be assumed to have. It can be shown, however, from results announced by Kripke in [1976] that if the relation is only assumed to be reflexive, or reflexive and symmetric but not also transitive, or reflexive and transitive but not also symmetric, then the wffs that are intensionally valid with respect to the relational structures in

question are not recursively enumerable; that is, the resulting semantics is then essentially incomplete. Whether the semantics is also incomplete for intensional validity with respect to the class of relational world systems in which the relation of accessibility is an equivalence relation, or, equivalently, in which it is universal between all the worlds in the system, has apparently not yet been determined (or at any rate not yet announced or published in the literature). However, because of its close similarity to Thomason's system **Q2** in [Thomason, 1969], the **S5** version of which Kripke in [1976] has claimed to be complete, we conjecture that it too is complete, i.e. that the set of wffs (of a given language) that are intensionally valid with respect to all relational world systems in which the relation of accessibility is universal is recursively enumerable. For convenience, we shall speak of the members of this set hereafter as being *intensionally valid simpliciter*; that is, we shall take the members of this set as being intensionally valid in the primary sense (while those that are valid otherwise are understood to be so in a secondary sense).

It is possible of course to give a secondary semantics in the sense in which quantification need not be with respect to *all* of the individual concepts in a given relational world system but only with respect to some non-empty set of such. (Cf. [Parks, 1976] where this gambit is employed—but in a semantics in which predicates have their extensions drawn in a given world from the restricted set of individual concepts and not from the possible objects.) As might be expected, completeness theorems are then forthcoming in the usual way even for classes of relational world systems in which the relation of accessibility is other than universal. Of course the question then arises as to the rationale for allowing *arbitrary* non-empty sets of individual concepts to be the basis for quantifying over such in any given relational world system. This question, moreover, is not really on a par with that regarding allowing arbitrary non-empty subsets of the set of possible worlds to be the referential basis of necessity (even where the relation of accessibility is universal); for in a framework in which the realm of intensionality is conceptually prior to its representation in terms of functions on possible worlds, the variability of the sets of possible world may in the end be analogous to the similar variability of the universes of discourses in standard (modal free) first-order logic. Such variability within the intensional realm itself, on the other hand, would seem to call for a different kind of explanation.

Thomason's semantical system **Q2** in [Thomason, 1969], it should be noted, differs from the above semantics for intensional validity *simpliciter* in requiring first that the set of existing objects of each possible world be non-empty, and, secondly, that although free individual variables range over the entire set of individual concepts, quantification in a given world is to be restricted to those individual concepts which denote objects that exist in that world. (Thomason also gives an 'outer domain' interpretation for improper definite descriptions which we can ignore here since definite de-

scriptions are not singular terms of the formal languages being considered.) Thus, clause (5) for assigning an intension to a quantified wff is replaced in **Q2** by:

- (5') $\text{Int}(\forall x\varphi, I, A) = \text{the } P \in 2^W \text{ such that for } w \in W, P(w) = 1 \text{ if for all individual concepts } f \text{ in } \langle W, R, E \rangle, \text{ if } f(w) \in E(w), \text{ then } \text{Int}(\varphi, I, A(f/x))(w) = 1.$

Intensional **Q2**-validity can now be defined as intensional validity with respect to all relational world systems in which (1) the set of objects existing in each world is non-empty, (2) the relation of accessibility is universal, and (3) quantification is interpreted as in clause(5'). According to Kripke [1976], the set of wffs (of a given language) that are intensionally **Q2**-valid is recursively enumerable. (Cf. [Bacon, 1980] for some of the history of these results and of an earlier erroneous claim by David Kaplan.) The completeness proof given in [Kamp, 1977], it should be noted, is not for **Q2**-validity (as might be thought from Kamp's remark that he is reconstructing Kripke's proof), but for a semantics in which individual concepts always denote only existing objects and in which the extension of a predicate's intension in a given world is drawn exclusively from the objects that exist in that world. Both conditions are too severe, however—at least from the point of view of the realm of intensional existence (an inexistence). In particular, whereas the rule of substitution:

$$\text{if } \models \varphi, \text{ then } \models \tilde{S}_{\psi}^{F(x_1, \dots, x_n)} \varphi \mid$$

is validated in both the semantics of intensional validity *simpliciter* and in the **Q2**-semantics, it is not validated in Kamp's more restricted semantics. Not all open wffs, in other words, represent predicate intensions, i.e. properties or relations in the logical sense, in Kamp's semantics—a result contrary to one of the basic motivations for the turn to intensionality.

9 INDIVIDUAL CONCEPTS AND THE ELIMINATION OF *DE RE* MODALITIES

One of the nice things about Thomason's **Q2**-semantics is that existence remains essentially a quantifier concept; that is, the definition of $E!$ given earlier remains in effect in the **Q2** semantics. This is not so of course in the semantics for intensional validity *simpliciter* where $E!$ would have to be introduced as a new intensional primitive (cf. [Bacon, 1980]). There would seem to be nothing really objectionable about doing so, however; or at least not from the point of view of the realm of intensional existence (and inexistence). Quantifying over all individual concepts, whether existent or in-existent—i.e. whether they denote objects that exist in the world in

question or not—can hardly be compared from this point of view with the different situation of quantifying over all possible objects in the semantics of a metaphysical necessity (as opposed to quantifying only over the objects that exist in the world in question in that metaphysical context).

One of the undesirable features of the **Q2**-semantics, however, is its validation of the wff

$$\exists x \Box E!(x)$$

which follows from the **Q2**-validity of

$$\Box \exists x \varphi \rightarrow \exists x \Box \varphi \quad \text{and} \quad \Box \exists x E!(x).$$

This situation can be easily rectified, of course, by simply rejecting the semantics of the latter wff, i.e. by not requiring the set of objects existing in each world of the **Q2**-semantics to be non-empty. The converse of the above conditional is not intensionally **Q2**-valid, incidentally, though both are intensionally valid in the primary sense; that is

$$(\Box \exists / \exists \Box) \exists x \Box \varphi \leftrightarrow \Box \exists x \varphi$$

is intensionally valid *simpliciter*. So of course is the Carnap–Barcan formula (and its converse), which also fails (in both directions) in the **Q2**-semantics; for this formula and its converse, it is well-known, is a consequence of the **S5** modal principles together with those of standard first-order predicate logic without identity (LPC). Of course, whereas every wff which is an instance of a theorem of LPC is intensionally valid in the primary sense, it is only their modal free-logic counterparts that are valid in the **Q2**-semantics. For example, whereas

$$\forall x \varphi \rightarrow \varphi(a/x)$$

is intensionally valid *simpliciter*, only its modal free-logic counterpart

$$\exists x \Box (a = x) \rightarrow [\forall x \varphi \rightarrow \varphi(a/x)]$$

is intensionally **Q2**-valid.

One rather important consequence of these results of the semantics for intensional validity in the primary sense, it should be noted, is the validation in this sense, of Von Wright's principle of predication (cf. [von Wright, 1951]) i.e. the principle (as restated here in terms of individual concepts) that if a property or relation in the logical sense is contingently predicable of the denotata of some individual concepts, then it is contingently predicable of the denotata of all individual concepts:

$$(\text{Pr}) \quad \exists x_1 \dots \exists x_n (\Diamond \varphi \wedge \Diamond \neg \varphi) \rightarrow \forall x_1 \dots \forall x_n (\Diamond \varphi \wedge \Diamond \neg \varphi).$$

The fact that (Pr) is intensionally valid *simpliciter* can be seen from the syntactic proof in [Broido, 1976] that

$$\text{LPC} + \mathbf{S5} + (\Box\exists/\exists\Box) \vdash (\text{Pr}).$$

That is, since every wff which is an axiom of $\text{LPC} + \mathbf{S5} + (\Box\exists/\exists\Box)$ is intensionally valid *simpliciter*, and *modus ponens* and universal modal generalisation preserve validity in this sense, then (Pr) is also intensionally valid *simpliciter*.

Now although not every wff which is an axiom of $\text{LPC} + (\Box\exists/\exists\Box)$ is valid in the **Q2**- semantics, nevertheless, it follows from the intensional validity of (Pr) in the primary sense that (Pr) is intensionally **Q2**-valid as well. To see this, assume that $E!$ is added as a new intensional primitive with the following clause added to the semantics of the preceding section:

$$\begin{aligned} \text{Int}(E!(a), I, A) &= \text{the } P \in 2^W \text{ such that for } w \in W, P(w) = 1 \\ &\text{iff } \text{Int}(a, I, A)(w) \in E(w). \end{aligned}$$

Then, where t translates each wff into its $E!$ restricted counterpart, i.e. where $t(\varphi) = \varphi$, for atomic wffs, $t(\neg\varphi) = \neg t(\varphi)$, $t(\varphi \rightarrow \psi) = (t(\varphi) \rightarrow t(\psi))$, $t(\Box\varphi) = \Box t(\varphi)$ and $t(\forall x\varphi) = \forall x(E!(x) \rightarrow t(\varphi))$, it can be readily seen that a wff φ is intensionally **Q2**-valid iff $[\exists x\Box E!(x) \rightarrow t(\varphi)]$ is intensionally valid *simpliciter*; and therefore if $t(\varphi)$ is intensionally valid *simpliciter*, then φ is intensionally **Q2**-valid. Now since

$$\exists x_1 \dots \exists x_n [\Diamond t(\varphi) \wedge \Diamond \neg t(\varphi)] \rightarrow \forall x_1 \dots \forall x_n [\Diamond t(\varphi) \wedge \Diamond \neg t(\varphi)]$$

is an instance of (Pr), it is intensionally valid *simpliciter*, and therefore so is

$$\begin{aligned} \exists x_1 \dots \exists x_n [E!(x_1) \wedge \dots \wedge E!(x_n) \wedge \Diamond t(\varphi) \wedge \Diamond \neg t(\varphi)] \rightarrow \\ \rightarrow \forall x_1 \dots \forall x_n [E!(x_1) \wedge \dots \wedge E!(x_n) \rightarrow \Diamond t(\varphi) \wedge \Diamond \neg t(\varphi)]. \end{aligned}$$

This last wff, however, is trivially equivalent to $t(\text{Pr})$. That is, $t(\text{Pr})$ is intensionally valid *simpliciter*, and therefore (Pr) is intensionally **Q2**-valid.

It is noteworthy, finally, that on the basis of $\text{LPC} + \mathbf{S5} + (\text{Pr}) + (\Box\exists/\exists\Box)$, [Broido, 1976] has shown that every *de re* wff is provably equivalent to a *de dicto* wff. Accordingly, since all of the assumptions or wffs essential to Broido's proof are intensionally valid *simpliciter*, it follows that every *de re* wff is eliminable in favour of only *de dicto* wffs (of modal degree ≤ 1) in the semantics of intensional validity in the primary sense.

THEOREM (De Re Elimination Theorem) For each *de re* wff φ , there is a *de dicto* wff ψ such that $(\varphi \leftrightarrow \psi)$ is intensionally valid *simpliciter*.

Kamp [1977] has shown, incidentally, that a *de re* elimination theorem also holds for the more restricted semantics in which individual concepts always denote existing objects and the extensions of predicate intensions

are always drawn exclusively from the objects existing in the world in question. (This theorem is in fact the basis of Kamp's completeness theorem for his semantics—and therefore perhaps also the basis for a similar proof of completeness for the semantics of intensional validity *simpliciter*.) Accordingly, since the logic of the **Q2**-semantics is intermediate between Kamp's semantics and the semantics of intensional validity in the primary sense, it is natural to conjecture that a similar *de re* elimination theorem also holds for intensional **Q2**-validity—though of course not one which depends on the consequences of $\text{LPC} + (\Box\exists/\exists\Box)$.

10 CONTINGENT IDENTITY

Identity in logical atomism, as we have already noted, does not stand for an atomic state of affairs; that is, despite its being represented by an atomic wff, an object's self-identity is part of the logical scaffolding of the world (which the world shares with every other possible world) and not part of the world itself. This is why identity is a non-contingent relation in logical atomism.

Identity in the realm of intensionality, on the other hand, is really not an identity of individual concepts but a world-bound relation of coincidence between these concepts. That is, as a relation in which individual concepts need not themselves be the same but only have the same denotation in a given world, 'identity' need not hold between the same individual concepts from world to world. This is why 'identity' can be a contingent relation from the intensional point of view.

In other words, whereas

$$\exists x\exists y(x = y \wedge \Diamond x \neq y), \quad \exists x\exists y(x \neq y \wedge \Diamond x = y)$$

are logically false from the point of view of the primary semantics for logical atomism, both can be true (and in fact must be true if there are at least two objects in the world) from the point of view of the realm of intensionality.

One argument in favour of contingent identity as a relation of coincidence between individual concepts is given in [Gibbard, 1975]. In Gibbard's example a clay statue named Goliath (hereafter represented by *a*) is said to be contingently identical with the piece of clay of which it is made and which is named Lumpl (hereafter represented by *b*). For convenience, we may suppose that *a* and *b* begin to exist at the same time; e.g. the statue is made first in two separate pieces which are then struck together 'thereby bringing into existence simultaneously a new piece of clay and a new statue' [Gibbard, 1975, p. 191]. Now although Goliath is Lumpl—i.e. ($a = b$) is true in the world in question—it is nevertheless possible that the clay is squeezed into a ball before it dries; and if that is done, then 'at that point . . . the statue Goliath would have ceased to exist, but the piece of clay Lumpl would still

exist in a new shape. Hence Lumpl would not be Goliath, even though both existed' [Gibbard, 1975]. That is, according to Gibbard, the wff

$$a = b \wedge \Diamond[a \neq b \wedge E!(a) \wedge E!(b)]$$

would be true in the world in question.

Contrary to Gibbard's claim, however, the above wff is not really a correct representation of the situation he describes. In particular, it is not true that Goliath exists in the world in which Lumpl has been squeezed into a ball. The correct description, in other words, is given by

$$a = b \wedge \Diamond[\neg E!(a) \wedge E!(b)],$$

which, since

$$a = b \rightarrow [E!(a) \leftrightarrow E!(b)]$$

is both intensionally valid *simpliciter* and intensionally **Q2**-valid, implies:

$$a = b \wedge \Diamond[a \neq b \wedge \neg E!(a) \wedge E!(b)]$$

and this wff in turn implies

$$\neg[a = b \rightarrow \Box a = b],$$

which is the conclusion Gibbard was seeking in any case. That is, the identity of Goliath with Lumpl, though true, is only contingently true.

Now it should be noted in this context that the thesis that proper names are rigid designators can be represented neither by

$$\exists x \Box(a = x) \quad \text{nor} \quad \Diamond \exists x \Box(a = x)$$

in the present semantics. For both wffs are in fact intensionally valid *simpliciter*, and the latter would be **Q2**-valid if we assumed that any individual concept expressed by a proper name always denotes an object which exists in at least one possible world—and yet, it is not required in either of these semantics that the individual concept expressed by a proper name is to denote the same object in every possible world. The question arises, accordingly, whether and in what sense Gibbard's example shows that the names 'Goliath' and 'Lumpl' are not rigid designators. For surely there is nothing in the way each name is introduced into discourse to indicate that its designation can change even when the object originally designated has continued to exist; and yet if it is granted that 'Goliath' and 'Lumpl' are rigid designators in the sense of designating the same object in every world in which it exists, then how is it that 'Goliath' can designate Lumpl in the one world where Goliath and Lumpl exist but not in the other where Lumpl but not Goliath exists? Doesn't the same object which 'Goliath' designates in the one world exist in the other? An answer to this problem is forthcoming, as we shall see, but from an entirely different perspective of the realm of intensionality; and indeed one in which identity is not a contingent relation either between objects or individual concepts.

11 QUANTIFIERS AS REFERENTIAL CONCEPTS

Besides the Platonic view of intensionality there is also the conceptualist view according to which concepts are not independently existing Platonic forms but cognitive capacities or related structures whose realisation in thought is what informs our mental acts with a predicable or referential nature. However, as cognitive capacities which may or may not be exercised on a given occasion, concepts, though they are not Platonic forms, are also neither mental images nor ideas in the sense of particular mental occurrences. That is, concepts are not objects or individuals but are rather unsaturated cognitive structures or dispositional abilities whose realisation in thought is what accounts for the referential and predicable aspects of particular mental acts.

Now the conceptual structures that account for the referential aspect of a mental act on this view are not the same as those that inform such acts with a predicable nature. A categorical judgement, for example, is a mental act which consists in the joint application of both types of concepts; that is, it is a mental event which is the result of the combination and mutual saturation of a referential concept with a predicable concept. Referential concepts, in other words, have a type of structure which is complementary to that of predicable concepts in that each can combine with the other in a kind of mental chemistry which results in a mental act having both a referential aspect and a predicable nature.

Referential concepts, it should be noted, are not developed initially as a form of reference to individuals *simpliciter* but are rather first developed as a form of reference to individuals of a given sort of kind. By a *sort* (or *sortal concept*) we mean in this context a type of common noun concept whose use in thought and communication is associated with certain identity criteria, i.e. criteria by which we are able to distinguish and count individuals of the kind in question. Typically, perceptual criteria such as those for shape, size and texture (hard, soft, liquid, etc.) are commonly involved in the application of such a concept; but then so are functional criteria (especially edibility) as well as criteria for the identification of natural kinds of things (animals, birds, fish, trees, plants, etc.) (cf. [Lyons, 1977, Vol. 2, Section 11.4]).

Though sortal concepts are expressed by common (count) nouns, not every common (count) noun, on the other hand, stands for a sort or kind in the sense intended here. Thus, e.g., whereas ‘thing’ and ‘individual’ are common (count) nouns, the concept of a thing or individual *simpliciter* is not associated in its use with any particular identity criteria, and therefore it is not a sortal concept in the sense intended here. Indeed, according to conceptualism, the concept of a thing or individual *simpliciter* has come to be constructed on the basis of the concept of a thing or individual of a certain sort (cf. [Sellars, 1963]). (It might be noted in this context, inci-

dentally, that while there are no explicit grammatical constructions which distinguish sortal common nouns from non-sortal common (count) nouns in the Indo-European language family, nevertheless there are 'classifier-languages'—e.g. Tzeltal, a Mayan language spoken in Mexico, Mandarin Chinese, Vietnamese, etc.—which do contain explicit and obligatory constructions involving sortal classifiers (cf. [Lyons, 1977]).

Reference to individuals of a given sort, accordingly, is not a form of restricted reference to individuals *simpliciter*; that is, referential concepts regarding these individuals are not initially developed as derived concepts based on a quantificational reference to individuals in general, but are themselves basic or underived sortal quantifier concepts. Thus, where S and T stand for sortal concepts, $(\forall xS)$, $(\forall yT)$, $(\exists zS)$, $(\exists xT)$, etc. can be taken on the view in question as basic forms of referential concepts whose application in thought enable us to refer to all S , all T , some S , some T , etc. respectively. For example, where S stands for the sort *man* and F stands for the predicable concept of *being mortal*, a categorical judgement that every man is mortal, or that some man is not mortal, can be represented by $(\forall xS)F(x)$ and $(\exists xS)\neg F(x)$, respectively. These formulas, it will be noted, are especially perspicuous in the way they represent the judgements in question as being the result of a combination and mutual saturation of a referential and predicable concept.

Though they are themselves basic or underived forms of referential concepts, sortal quantifiers are nevertheless a special type of common (count) noun quantifier—including, of course, the ultimate common (count) noun quantifiers $\forall x$ and $\exists x$ (as applied with respect to a given individual variable x). Indeed, the latter, in regard to the referential concepts they represent, would be more perspicuous if written out more fully as $(\forall x \text{ Individual})$ and $(\exists x \text{ Individual})$, respectively. The symbols $\forall$ and $\exists$, in other words, do not stand in conceptualism for separate cognitive elements but are rather 'incomplete symbols' occurring as parts of common (count) noun quantifiers. For convenience, however, we shall continue to use the standard notation $\forall x$ and $\exists x$ as abbreviations of these ultimate common (count) noun quantifiers.

12 SINGULAR REFERENCE

As represented by common (count) noun quantifiers, referential concepts are indeed complementary to predicable concepts in exactly the way described by conceptualism; that is, they are complementary in the sense that when both are applied together it is their combination and mutual saturation in a kind of mental chemistry which accounts for the referential and predicable aspects of a mental act. It is natural, accordingly, that a parallel interpretation should be given for the referential concepts underlying the use of singular terms.

Such an interpretation, it will be observed, is certainly a natural concomitant of Russell's theory of definite descriptions—or rather of Russell's theory somewhat modified. Where S , for example, is a common (count) noun, including the ultimate common (count) noun 'individual', the truth-conditions for a judgement of the form

1. the S wh. is F is G

will be semantically equivalent in conceptualism to those for the wff

2. $(\exists xS)[(\forall yS)(F(y) \leftrightarrow y = x) \wedge G(x)]$

if, in fact, the definite description is being used in that judgement with an existential presupposition. If it is not being so used, however, then the truth-conditions for the judgement are semantically equivalent to

3. $(\forall xS)[(\forall yS)[F(y) \leftrightarrow y = x] \rightarrow G(x)]$

instead. Note however that despite the semantical equivalence of one of these wffs with the judgement in question, neither of them can be taken as a direct representation of the cognitive structure of that judgement. Rather, where ' S wh F '; abbreviates ' S wh. is F ', a more perspicuous representation of the judgement can be given either by

4. $(\exists_1 xS \text{ wh } F)G(x)$

or

5. $(\forall_1 xS \text{ wh } F)G(x)$

respectively, depending on whether the description is being used with or without an existential presupposition. (The 'incomplete' quantifier symbols $\exists_1$ and $\forall_1$ are, of course, understood here in such a way as to render (4) and (5) semantically equivalent to (2) and (3), respectively.) The referential concept which underlies using the definite description with an existential presupposition, in other words, is the concept represented by $(\exists_1 xS \text{ wh } F)$; and of course the referential concept which underlies using the description without an existential presupposition is similarly represented by $(\forall_1 xS \text{ wh } F)$.

Now while definite descriptions are naturally assimilated to quantifiers, proper names are in turn naturally assimilated to sortal common nouns. For just as the use of a sortal is associated in thought with certain identity criteria, so too is the introduction and use of a proper name (whose identity criteria are provided in part by the most specific sortal associated with the introduction of that name and to which the name is thereafter subordinate). In this regard, the referential concept underlying the use of a proper name is determined by the identity criteria associated with that name's introduction into discourse.

On this interpretation, accordingly, the referential concept underlying the use of a proper name corresponds to the referential concept underlying the use of a sortal common noun; that is, both are to be represented by sortal quantifiers (where 'sortal' is now taken to encompass proper names as well). The only difference between the two is that when such a quantifier contains a proper name, it is always taken to refer to at most a single individual. Thus, if in someone's statement that Socrates is wise, 'Socrates' is being used with an existential presupposition, then the statement can be represented by

$$(\exists x \text{ Socrates})(x \text{ is wise}).$$

If 'Socrates' is being used without an existential presupposition, on the other hand, then the statement can be represented by

$$(\forall x \text{ Socrates})(x \text{ is wise})$$

instead. That is, the referential concepts underlying using 'Socrates' with and without an existential presupposition can be represented by $(\exists x \text{ Socrates})$ and $(\forall x \text{ Socrates})$, respectively. (Such a quantifier interpretation of the use of proper names will also explain, incidentally, why the issue of scope is relevant to the use of a proper name in contexts involving the expression of a propositional attitude.)

Now without committing ourselves at this point as to the sense in which conceptualism can allow for the development of alethic modal concepts, i.e. modal concepts other than those based upon a propositional attitude, it seems clear that the identity criteria associated with the use of a proper name do not change when that name is used in such a modal context. That is, the demand that we need a criterion of identity across the possible worlds associated with such a modality in order to determine whether a proper name is a rigid designator or not is without force in conceptualism since, in fact, such a criterion is already implicit in the use of a proper name. In other words, where S is a proper name, we can take it as a conceptual truth that the identity criteria associated with the use of S (1) always picks out at most one object and (2) that it is the same object which is so picked out whenever it exists:

$$(PN) \quad (\forall xS)[\Box(\forall yS)(y = x) \wedge \Box(E!(x) \rightarrow (\exists yS)(x = y))].$$

Nothing in this account of proper names conflicts, it should be noted, with Gibbard's example of the statue Goliath which is identical with Lump1, the piece of clay of which it consists, during the time of its existence, but which ceases to be identical with Lump1 because it ceases to exist when Lump1 is squeezed into a ball. Both names, in other words, can be taken as rigid designators in the above sense without resulting in a contradiction in the situation described by Gibbard. Where S and T , for example, are proper

name sortals for ‘Goliath’ and ‘Lumpl’, respectively, the situation described by Gibbard is consistently represented by the following wff:

$$(\exists xS)(\exists yT)(x = y) \wedge \Diamond(\exists yT)(\forall xS)(x \neq y).$$

That is, whereas the identity criteria associated with ‘Goliath’ and ‘Lumpl’ enable us to pick out the same object in the original world or time in question, it is possible that the criteria associated with ‘Lumpl’ enable us to pick out an object identifiable as Lumpl in a world or time in which there is no object identifiable as Goliath. In this sense, conceptualism is compatible with the claim that there can be contingent identities containing only proper names—even though proper names are rigid designators in the sense of satisfying (PN).

It does not follow, of course, that identity is a contingent relation in conceptualism; and, indeed, quite the opposite is the case. That is, since *reference in conceptualism is directly to objects*, albeit mediated by referential concepts, it is a conceptual truth to say that an object cannot but be the object that it is or that one object cannot be identical with another. In other words, the following wffs:

$$\forall x\forall y(x = y \rightarrow \Box x = y), \quad \forall x\forall y(x \neq y \rightarrow \Box x \neq y)$$

are to be taken as valid theses of conceptualism. This result is the complete opposite, needless to say, from that obtained *on the Platonic view* where *reference is directly to individual concepts* (as independently existing platonic forms) and only indirectly to the objects denoted by these concepts in a given possible world.

13 CONCEPTUALISM AND TENSE LOGIC

As forms of conceptual activity, thought and communication are inextricably temporal phenomena, and to ignore this fact in the semantics of a formal representation of such activity is to court possible confusion of the Platonic with the conceptual view of intensionality. Propositions, for example, on the conceptual view, are not abstract entities existing in a platonic realm independently of all conceptual activity. Rather, according to conceptualism, they are really conceptual constructs corresponding to the truth-conditions of our temporally located assertions; and on the present level of analysis where propositional attitudes are not being considered, their status as constructs can be left completely in the metalanguage.

What is also a construction, but which should not be left to the metalanguage, are certain cognitive schemata characterising our conceptual orientation in time and implicit in the form and content of our assertions as mental acts. These schemata, whether explicitly recognised as such or not, are usually represented or modelled in terms of a tenseless idiom (such as

our set-theoretic metalanguage) in which reference can be made to moments or intervals of time (as individuals of a special type); and for most purposes such a representation is quite in order. But to represent them only in this way in a context where our concern is with a perspicuous representation of the form of our assertions as mental acts might well mislead us into thinking that the schemata in question are not essential in conceptualism to the form and content of an assertion after all—the way they are not essential to the form and content of a proposition on the Platonic view. Indeed, even though the cognitive schemata in question can be modelled in terms of a tenseless idiom of moments or intervals of time (as in fact they will be in our set-theoretic metalanguage), they are themselves the conceptually prior conditions that lead to the construction of our referential concepts for moments or intervals of time, and therefore of the very tenseless idiom in which they are subsequently modelled. In this regard, no assumption need be made in conceptualism about the ultimate nature of moments or intervals of time, i.e. whether such entities are really independently existing individuals or only constructions out of the different events that actually occur.

Now since what the temporal schemata implicit in our assertions fundamentally do is enable us to orientate ourselves in time in terms of the distinction between the past, the present, and the future, a more appropriate or perspicuous representation of these schemata is one based upon a system of quantified tense logic containing at least the operators $\mathcal{P}, \mathcal{N}, \mathcal{F}$ for ‘it was the case that’, ‘it is now the case that’, and ‘it will be the case that’, respectively. As applied in thought and communication, what these operators correspond to is our ability to refer to what was the case, what is now the case, and what will be the case—and to do so, moreover, without having first to construct referential concepts for moments or intervals of time.

Keeping our analysis as simple as possible, accordingly, let us now understand a language to consist of symbols for common (count) nouns, including always one for ‘individual’, as well as proper names and predicates. Where L is such a language, the atomic wffs of L are expressions of the form $(x = y)$ and $F(x_1, \dots, x_n)$, where $x, y, x_1, \dots, x_n$ are variables and F is an n -place predicate in L . The wffs of L are then the expressions in every set K containing the atomic wffs of L and such that $\neg\varphi, \mathcal{P}\varphi, \mathcal{N}\varphi, \mathcal{F}\varphi, (\varphi \rightarrow \psi), (\forall xS)\varphi$ are all in K whenever $\varphi, \psi \in K, x$ is an individual variable and S is either a proper name or a common (count) noun symbol in L . As already noted, where S is the symbol for ‘individual’, we take $\forall x\varphi$ to abbreviate $(\forall xS)\varphi$, and similarly $\exists x\varphi$ abbreviates $\neg(\forall xS)\neg\varphi$.

In regard to a set-theoretic semantics for these wffs, let us retain the notion of a relational world system $\langle W, R, E \rangle$ already defined, but with the understanding that the members of W are now to be the moments of a *local time* (*Eigenzeit*) rather than complete possible worlds, and that the

relation R of accessibility is the earlier-than relation between the moments of that local time. The only constraint imposed by conceptualism on the structure of R is that it be a linear ordering of W , i.e. that R be asymmetric, transitive and connected in W . This constraint is based upon the implicit assumption that a local time is always the local time of a continuant.

There is nothing in set theory itself, it should be noted, which directly corresponds to the unsaturated nature of concepts as cognitive capacities; and for this reason we shall once again follow the Carnapian approach and represent concepts as functions from the moments of a local time to the classes of objects falling under the concepts at those times. Naturally, on this approach one and the same type of function will be used to represent the concepts underlying the use of common (count) nouns, proper names, and one-place predicates—despite the conceptual distinctions between them and in the way they account for different aspects of a mental act.

Accordingly, where L is a language and $\langle W, R, E \rangle$ is a relational world system, we shall now understand an *interpretation* for L based upon $\langle W, R, E \rangle$ to be a function I on L such that (1) for each n -place predicate F^n in L , $I(F^n)$ is an n -place predicate intension in $\langle W, R, E \rangle$; (2) for each common (count) noun symbol S in L , $I(S)$ is a one-place predicate intension in $\langle W, R, E \rangle$; and for the symbol S for 'individual' in particular, $I(S)(w) = E(w)$, for all $w \in W$; and (3) for each proper name S in L , $I(S)$ is a one-place predicate intension in $\langle W, R, E \rangle$ such that for some $d \in \cup_{w \in W} E(w)$, $I(S)(w) \subseteq \{d\}$, for all $w \in W$. (Note that at any given time $w \in W$, nothing need, in fact, be identifiable by means of the identity criteria associated with the use of a proper name—though if anything is so identifiable, then it is always the same individual. In this way we trivially validate the thesis (PN) of the preceding section that proper names are rigid designators with respect to the modalities analysable in terms of time.)

By a *referential assignment* in a relational world system $\langle W, R, E \rangle$, we now understand a function A which assigns to each variable x a member of $\cup_{w \in W} E(w)$, hereafter called the *realia* of $\langle W, R, E \rangle$. *Realia*, of course, are the objects that exist at some time or other of the local time in question. Referential concepts, at least in the semantics formulated below, do not refer directly to *realia*, but only indirectly (of which more anon); and in this regard *realia* are in tense logic what *possibilia* are in modal logic.

Finally, where t is construed as *the present moment* of a local time $\langle W, R, E \rangle$, i.e. $t \in W$, A is a referential assignment in $\langle W, R, E \rangle$ and I is an interpretation for a language L based on $\langle W, R, E \rangle$, we recursively define with respect to I and A *the proposition* (or intension in the sense of the truth-conditions) *expressed by a wff φ of L when part of an assertion made at t* as follows:

1. $\text{Int}_t(x = y, I, A) =$ the $P \in 2^W$ such that for $w \in W$, $P(w) = 1$ iff $A(x) = A(y)$;
2. $\text{Int}_t(F^n(x_1, \dots, x_n), I, A) =$ the $P \in 2^W$ such that for $w \in W$, $P(w) = 1$ iff $\langle A(x_1), \dots, A(x_n) \rangle \in I(F^n)(w)$;
3. $\text{Int}_t(\neg\varphi, I, A) =$ the $P \in 2^W$ such that for $w \in W$, $P(w) = 1$ iff $\text{Int}_t(\varphi, I, A)(w) = 0$;
4. $\text{Int}_t(\varphi \rightarrow \psi, I, A) =$ the $P \in 2^W$ such that for $w \in W$, $P(w) = 1$ iff $\text{Int}_t(\varphi, I, A)(w) = 0$ or $\text{Int}_t(\psi, I, A)(w) = 1$;
5. $\text{Int}_t((\forall xS)\varphi, I, A) =$ the $P \in 2^W$ such that for $w \in W$, $P(w) = 1$ iff for all $d \in E(w)$, if $d \in I(S)(w)$, then $\text{Int}_t(\varphi, I, A(d/x))(w) = 1$;
6. $\text{Int}_t(\mathcal{P}\varphi, I, A) =$ the $P \in 2^W$ such that for all $w \in W$, $P(w) = 1$ iff $\text{Int}_t(\varphi, I, A)(u) = 1$, for some u such that uRw ;
7. $\text{Int}_t(\mathcal{N}\varphi, I, A) =$ the $P \in 2^W$ such that for all $w \in W$, $P(w) = 1$ iff $\text{Int}_t(\varphi, I, A)(t) = 1$; and
8. $\text{Int}_t(\mathcal{F}\varphi, I, A) =$ the $P \in 2^W$ such that for all $w \in W$, $P(w) = 1$ iff $\text{Int}_t(\varphi, I, A)(u) = 1$, for some u such that wRu .

The double-indexing involved in this semantics and critically used in clause (7) is to account for the role of the now-operator. It was first given in [Kamp, 1971] and, of course, is particularly appropriate for conceptualism's concern with the semantics of assertions as particular mental acts. That is, as constructed in terms of the truth-conditions for assertions, propositions on the conceptualist's view of intensionality differ from those of the Platonist in being bound to the time at which the assertion in question occurs. For the Platonist, propositions exist independently of time, and therefore of the truth-conditions for assertions as well.

In regard to truth and validity, we shall say, relative to an interpretation I and referential assignment A in a local time $\langle W, R, E \rangle$, that a wff φ of the language in question is *true* if $\text{Int}_t(\varphi, I, A)(t) = 1$, where t is the present moment of the local time $\langle W, R, E \rangle$. The wff φ is said to be *valid* or *tense-logically true*, on the other hand, if for all local time systems $\langle W, R, E \rangle$, all $t \in W$, all referential assignments A in $\langle W, R, E \rangle$, and all interpretations I for a language of which φ is a wff, $\text{Int}_t(\varphi, I, A)(t) = 1$.

A completeness theorem is forthcoming for this semantics, but we shall not concern ourselves with establishing one in the present essay—especially since the overall logic is rather weak or minimal in the way it accounts for our conceptual orientation in time. Instead, let us briefly examine the problem of referring to *realia* in general, and in particular to past or future objects—i.e. the problem of how the conceptual structure of such a minimal system can either account for such reference or lead to a conceptual development where such an account can be given.

14 THE PROBLEM OF REFERENCE TO PAST AND FUTURE OBJECTS

Our comparison of the status of *realia* in tense logic with *possibilia* in modal logic is especially appropriate, it might be noted, insofar as quantificational reference to either is said to be feasible only *indirectly*—i.e. through the occurrence of a quantifier within the scope of a modal or tense operator (cf. [Prior, 1967, Chapter 8]). The reference to a past individual in ‘Someone did exist who was a King of France’, for example, can be accounted for by the semantics of $\mathcal{P}(\exists xS)(\exists yT)(x = y)$, where S and T are sortal common noun symbols for ‘person’ and ‘King of France’, respectively. What is apparently not feasible about a *direct* quantificational reference to such objects, on this account, is our present inability to actually confront and apply the relevant identity criteria to objects which do not now exist.

A present ability to identify past or future objects of a given sort, however, is not the same as the ability to actually confront and identify those objects in the present; that is, our existential inability to do the latter is not the same as, and should not be confused with, what is only presumed to be our inability to directly refer to past or future objects. Indeed, the fact is that we can and do make *direct reference* to *realia*, and to past and future objects in particular, and that we do so not only in ordinary discourse but also, and especially, in most if not all of our scientific theories. The real problem is not that we cannot directly refer to past and future objects, but rather how it is that conceptually we come to do so.

One explanation of how this comes to be can be seen in the analysis of the following English sentences:

1. There did exist someone who is an ancestor of everyone now existing.
2. There will exist someone who will have everyone now existing as an ancestor.

Where S is a sortal common noun symbol for ‘person’ and $R(x, y)$ is read as ‘ x is an ancestor of y ’, it is clear that (1) and (2) cannot be represented by:

3. $\mathcal{P}(\exists xS)(\forall yS)R(x, y)$
4. $\mathcal{F}(\exists xS)(\forall yS)R(y, x)$.

For what (3) and (4) represent are the different sentences:

5. There did exist someone who was an ancestor of everyone then existing.
6. There will exist someone who will have everyone then existing as an ancestor.

Of course, if referential concepts that enabled us to refer directly to past and future objects were already available, then the obvious representation of (1) and (2) would be:

$$7. (\exists x \text{ Past-}S)(\forall yS)R(x, y)$$

$$8. (\exists x \text{ Future-}S)(\forall yS)\mathcal{F}R(y, x)$$

where ‘Past-’ and ‘Future-’ are construed as common noun modifiers. (We assume here that the relational ancestor concept is such that x is an ancestor of y only at those times when either y exists and x did exist, though x need not still exist at the time in question, or when x has continued to exist even though y has ceased to exist. When y no longer exists as well as x , we say that x *was* an ancestor of y ; and where y has yet to exist, we say that x *will be* an ancestor of y .)

Now although these last analyses are not available in the system of tense logic formulated in the preceding section, nevertheless semantical equivalences for them are. In this regard, note that although the indirect references to past and future objects in (3) and (4) fail to provide adequate representations of (1) and (2), the same indirect references followed by the now-operator succeed in capturing the direct references given in (7) and (8):

$$9. \mathcal{P}(\exists xS)\mathcal{N}(\forall yS)R(x, y)$$

$$10. \mathcal{F}(\exists xS)\mathcal{N}(\forall yS)\mathcal{F}R(y, x).$$

In other words, at least relative to any present tense context, we can in general account for direct reference to past and future objects as follows:

$$(\forall x \text{ Past-}S)\varphi \leftrightarrow \neg\mathcal{P}\neg(\forall xS)\mathcal{N}\varphi$$

$$(\forall x \text{ Future-}S)\varphi \leftrightarrow \neg\mathcal{F}\neg(\forall xS)\mathcal{N}\varphi.$$

These equivalences, it should be noted, cannot be used other than in a present tense context; that is, the above use of the now-operator would be inappropriate when the equivalences are stated within the scope of a past- or future-tense operator, since in that case the direct reference to past or future objects would be from a point of time other than the present. Formally, what is needed in such a case is the introduction of so-called ‘backwards-looking’ operators, such as ‘then’, which can be correlated with occurrences of past or future tense operators within whose scope they lie and which semantically evaluate the wffs to which they are themselves applied in terms of the past or future times already referred to by the tense operators with which they are correlated (cf. [Vlach, 1973] and [Saarinen, 1976]). Backwards-looking operators, in other words, enable us to conceptually return to a past or future time already referred to in a given context in the same way that the now-operator enables us to return to the present. In that regard, their role in the cognitive schemata characterising our conceptual orientation in time

and implicit in each of our assertions is essentially a projection of the role of the now-operator.

We shall not formulate the semantics of these backwards-looking operators here, however; but we note that with their formulation equivalences of the above sort can be established for all contexts of tense logic, past and future as well as present. In any case, it is clear that the fact that conceptualism can account for the development of referential concepts that enable us to refer directly to past or future objects is already implicit in the fact that such references can be made with respect to the present alone. for this already shows that whereas the reference is direct at least in effect, nevertheless the application of any identity criteria associated with such reference will itself be indirect, and in particular, not such as to require a present confrontation, even if only in principle, with a past or future object.

15 TIME AND MODALITY

One important feature of the cognitive schemata characterising our conceptual orientation in time and represented in part by quantified tense logic, according to conceptualism, is the capacity they engender in us to form modal concepts having material content. Indeed, some of the first such modal concepts every to be formulated in the history of thought are based precisely upon the very distinction between the past, the present, and the future which is contained in these schemata. For example, the Megaric logician Diodorus is reported as having argued that the possible is that which either is or will be the case, and that therefore the necessary is that which is and always will be the case (cf. [Prior, 1967, Chapter 2]):

$$\Diamond^f \varphi = df \varphi \vee \mathcal{F}\varphi, \quad \Box^f \varphi = df \varphi \wedge \neg \mathcal{F}\neg \varphi.$$

Aristotle, on the other hand, included the past as part of what is possible; that is, for Aristotle the possible is that which either was, is, or will be the case (in what he assumed to be the infinity of time), and therefore the necessary is what is always the case (cf. [Hintikka, 1973]):

$$\Diamond^t \varphi = df \mathcal{P}\varphi \vee \varphi \vee \mathcal{F}\varphi, \quad \Box^t \varphi = df \neg \mathcal{P}\neg \varphi \wedge \varphi \wedge \neg \mathcal{F}\neg \varphi.$$

Both Aristotle and Diodorus, it should be noted, assumed that time is real and not ideal—as also does the socio-biologically based conceptualism being considered here. The temporal modalities indicated above, accordingly, are in this regard intended to be taken as material or metaphysical modalities (of a conceptual realism); and, indeed, they serve this purpose rather well, since in fact they provide a paradigm by which we might better understand what is meant by a material or metaphysical modality. In particular, not only do these modalities contain an explanatory, concrete

interpretation of the accessibility relation between possible worlds (now reconstrued as momentary states of the universe), but they also provide a rationale for the secondary semantics of a metaphysical necessity—since clearly not every possible world (of a given logical space) need ever actually be realised in time (as a momentary state of the universe). Moreover, the fact that the semantics (as considered here) is concerned with concepts and not with independently real material properties and relations (which may or may not correspond to some of these concepts but which can in any case also be considered in a supplementary semantics of conceptual realism) also explains why predicates can be true of objects at a time when those objects do not exist. For concepts, such as that of being an ancestor of everyone now existing, are constructions of the mind and can in that regard be applied to past or future objects no less so than to presently existing objects. In addition, because the intellect is subject to the closure conditions of the laws of compositionality for systematic concept-formation, there is no problem in conceptualism regarding the fact that a concept can be constructed corresponding to every open wff—thereby validating the rule of substitution of wffs for predicate letters.

As a paradigm of a metaphysical modality, on the other hand, one of the defects of Aristotle's notion of necessity is its exclusion of certain situations that are possible in special relativity. For example, relative to the present of a given local time, a state of affairs can come to have been the case, according to special relativity, without its ever actually being the case (cf. [Putnam, 1967]). That is, where $\mathcal{FP}\varphi$ represents φ 's coming (future) to have been (past) the case, and $\neg\Diamond^t\varphi$ represents φ 's never actually being the case, the situation envisaged in special relativity might be thought to be represented by:

$$\mathcal{FP}\varphi \wedge \neg\Diamond^t\varphi.$$

This conjunction, however, is incompatible with the linearity assumption of the local time in question; for on the basis of that assumption

$$\mathcal{FP}\varphi \rightarrow \mathcal{P}\varphi \vee \varphi \vee \mathcal{F}\varphi$$

is tense-logically true, and therefore $\mathcal{FP}\varphi$, the first conjunct, implies $\Diamond^t\varphi$, which contradicts the second conjunct, $\neg\Diamond^t\varphi$. The linearity assumption, moreover, cannot be given up without violating the notion of a local time or that of a continuant upon which it is based; and the notion of a continuant, as already indicated, is a fundamental construct of conceptualism. In particular, the notion of a continuant is more fundamental even than that of an event, which (at least initially) in conceptualism is always an occurrence in which one or more continuants are involved. Indeed, the notion of a continuant is even more fundamental in a socio-biologically based conceptualism than the notion of the self as a centre of conceptual activity, and it

is in fact one of the bases upon which the tense-logical cognitive schemata characterising our conceptual orientation in time are constructed.

This is not to say, on the other hand, that in the development of the concept of a self as a centre of conceptual activity we do not ever come to conceive of the ordering of events from perspectives other than our own. Indeed, by a process which Jean Piaget calls *decentering*, children at the stage of concrete operational thought (7–11 years) develop the ability to conceive of projections from their own positions to that of others in their environment; and subsequently, by means of this ability, they are able to form operational concepts of space and time whose systematic co-ordination results essentially in the structure of projective geometry. Spatial considerations aside, however, and with respect to time alone, the cognitive schemata implicit in the ability to conceive of such projections can be represented in part by means of tense operators corresponding to those already representing the past and the future as viewed from one's own local time. That is, since the projections in question are to be based on actual causal connections between continuants, we can represent the cognitive schemata implicit in such projections by what we shall here call *causal tense operators*, viz. $\mathcal{P}_c$ for 'it causally was the case that' and $\mathcal{F}_c$ for 'it casually will be the case that'. Of course, the possibility in special relativity of a state of affairs coming to have been the case without its ever actually being the case is a possibility that should be represented in terms of these operators and not in terms of those characterising the ordering of events within a single local time.

Semantically, in other words, the causal tense operators go beyond the standard tenses by requiring us to consider not just a single local time but a causally connected system of such local times. In this regard, the causal connections between the different continuants upon which such local times are based can simply be represented by a signal relation between the momentary states of those continuants—or rather, and more simply yet, by a signal relation between the moments of the local times themselves, so long as we assume that the sets of moments of different local times are disjoint. (This assumption is harmless if we think of a moment of a local time as an ordered pair one constituent of which is the continuant upon which that local time is based.) The only constraint that should be imposed on such a signal relation is that it be a strict partial ordering, i.e. transitive and asymmetric. Of course, since we assume that there is a causal connection from the earlier to the later momentary states of the same continuant, we shall also assume that the signal relation contains the linear temporal ordering of the moments of each local time in such a causally connected system. (Cf. [Carnap, 1958, Sections 49–50], for one approach to the notion of a causally connected system of local times.) Needless to say, but such a signal relation provides yet another concrete interpretation of the accessibility relation between possible worlds (reconstrued as momentary

states of the universe); and it will be in terms of this relation that the semantics of the causal tense operators will be given.

Accordingly, by a *system of local times* we shall understand a pair $\langle K, S \rangle$ such that (1) K is a non-empty set of relational world systems $\langle W, R, E \rangle$ for which (a) R is a linear ordering of W and (b) for all $\langle W', R', E' \rangle \in K$, if $\langle W, R, E \rangle \neq \langle W', R', E' \rangle$, then W and W' are disjoint; and (2) S is a strict partial ordering of $\{w : \text{for some } \langle W, R, E \rangle \in K, w \in W\}$ and such that for all $\langle W, R, E \rangle \in K, R \subseteq S$. Furthermore, if $\langle W, R, E \rangle, \langle W', R', E' \rangle \in K, t \in W$, and $t' \in W'$, then t is said to be *simultaneous* with t' in $\langle K, S \rangle$ iff neither tSt' nor $t'St$; and t is said to *coincide with* t' iff for all $\langle W'', R'', E'' \rangle \in K$ and all $w \in W''$, (1) w is simultaneous with t in $\langle K, S \rangle$ iff w is simultaneous with t' in $\langle K, S \rangle$, and (2) tSw if $t'Sw$.

Now a system $\langle K, S \rangle$ of local times is said to be *causally connected* iff for all $\langle W, R, E \rangle, \langle W', R', E' \rangle \in K$, (1) for all $t \in W, t' \in W'$, if t coincides with t' in $\langle K, S \rangle$, then $E(t) = E'(t')$, i.e. the same objects exist at coinciding moments of different local times; and (2) for all $t, w \in W$, all $t', w' \in W'$, if t is simultaneous with t' in $\langle K, S \rangle$, w is simultaneous with w' in $\langle K, S \rangle$, tRw and $t'R'w'$, then $\{\langle t, u \rangle : tRu \wedge uRw\} \cong \{\langle t', u \rangle : t'R'u \wedge uR'w'\}$; i.e. the structure of time is the same in any two local intervals whose end-points are simultaneous in $\langle K, S \rangle$.

Note that although the relation of coincidence in a causally connected system is clearly an equivalence relation, the relation of simultaneity, at least in special relativity, need not even be transitive. This will, in fact, be a consequence of the principal assumption of special relativity, viz. that the signal relation S of a causally connected system $\langle K, S \rangle$ has a *finite limiting velocity*; i.e. for all $\langle W, R, E \rangle, \langle W', R', E' \rangle \in K$ and all $w \in W$, if w does not coincide in $\langle K, S \rangle$ with any moment of W' , then there are moments u, v of W' such that $uR'v$ and yet w is simultaneous with both u and v in $\langle K, S \rangle$ (cf. [Carnap, 1958]). It is, of course, because of this assumption that a state of affairs can come (causal future) to have been (causal past) the case without its ever actually being the case (in the local time in question).

Finally, where

$$[t]_{\langle K, S \rangle} = \{t' : t' \text{ coincides with } t \text{ in } \langle K, S \rangle\},$$

$$W_{\langle K, S \rangle} = \{[t]_{\langle K, S \rangle} : \text{for some } \langle W, R, E \rangle \in K, t \in W\},$$

$$R_{\langle K, S \rangle} = \{\langle [t]_{\langle K, S \rangle}, [w]_{\langle K, S \rangle} \rangle : tSw\},$$

$$E_{\langle K, S \rangle} = \{\langle [t]_{\langle K, S \rangle}, E(t) \rangle : \text{for some } W, R, \langle W, R, E \rangle \in K \text{ and } t \in W\},$$

then $\langle W_{\langle K, S \rangle}, R_{\langle K, S \rangle}, E_{\langle K, S \rangle} \rangle$ is a relational world system (in which every theorem of **S4** is validated). Accordingly, if I is an interpretation for a language L based on $\langle W_{\langle K, S \rangle}, R_{\langle K, S \rangle}, E_{\langle K, S \rangle} \rangle$, A is a referential assignment in $\langle W_{\langle K, S \rangle}, R_{\langle K, S \rangle}, E_{\langle K, S \rangle} \rangle$, $\langle W, R, E \rangle \in K$, and $t \in W$, then we recursively define with respect to I and A the *proposition expressed by a* wff φ of L *when part of an assertion made at* t (as the present of the local time

$\langle W, R, E \rangle$ which is causally connected in the system $\langle K, S \rangle$, in symbols $\text{Int}_t(\varphi, I, A)$, exactly as before (in Section 13), except for the addition of the following two clauses:

9. $\text{Int}_t(\mathcal{P}_c\varphi, I, A) =$ the $P \in 2^W$ such that for $w \in W$, $P(w) = 1$ iff there are a local time $\langle W', R', E' \rangle \in K$ and moments $t', u \in W'$ such that t is simultaneous with t' in $\langle K, S \rangle$, uSw , and $\text{Int}_{t'}(\varphi, I, A)(u) = 1$; and
10. $\text{Int}_t(\mathcal{F}_c\varphi, I, A) =$ the $P \in 2^W$ such that for $w \in W$, $P(w) = 1$ iff there are a local time $\langle W', R', E' \rangle \in K$ and moments $t', u \in W'$ such that t is simultaneous with t' in $\langle K, S \rangle$, wSu and $\text{Int}_{t'}(\varphi, I, A)(u) = 1$.

Except for an invariance with respect to the added parameter $\langle K, S \rangle$, validity or tense-logical truth is understood to be defined exactly as before. It is clear of course that although

$$\mathcal{P}\varphi \rightarrow \mathcal{P}_c\varphi, \quad \mathcal{F}\varphi \rightarrow \mathcal{F}_c\varphi$$

are valid, their converses can be invalidated in a causally connected system which has the finite limiting velocity. On the other hand, were we to exclude such systems (as was done in classical physics) and validate the converse of the above wffs as well (as perhaps is still implicit in our common sense framework), then, of course, the causal tense operators would be completely redundant (which perhaps explains why they have no counterparts in natural language). It should perhaps be noted here that unlike the cognitive schemata of the standard tense operators whose semantics is based on a single local time, those represented by the causal tense operators are not such as must be present in one form or another in every act of thought. That is, they are derived schemata, constructed on the basis of those de-centering abilities whereby we are able to conceive of the ordering of events from a perspective other than our own. Needless to say, but the importance and real significance of these derived schemata was unappreciated until the advent of special relativity.

One important consequence of the divergence of the causal from the standard tense operators is the invalidity of

$$\mathcal{F}_c\mathcal{P}_c\varphi \rightarrow \mathcal{P}_c\varphi \vee \varphi \vee \mathcal{F}_c\varphi$$

and therefore the consistency of

$$\mathcal{F}_c\mathcal{P}_c\varphi \wedge \neg\Diamond^t\varphi.$$

Unlike its earlier counterpart in terms of the standard tenses, this last wff of course is the appropriate representation of the possibility in special relativity of a state of affairs coming (in the causal future) to have been the case (in the causal past) without its ever actually being the case (in a given local

time). Indeed, not only can this wff be true at some moment of a local time of a causally connected system, but so can the following wff:

$$[\mathcal{P}_c \Diamond^t \varphi \vee \mathcal{F}_c \Diamond^t \varphi] \wedge \neg \Diamond^t \varphi.$$

Quantification over *realia*, incidentally, finds further justification in special relativity. For just as some states of affairs can come to have been the case (in the causal past of the causal future) without their actually ever being the case, so too there can be things that exist only in the past or future of our own local time, but which nevertheless might exist in a causally connected local time at a moment which is simultaneous with our present. In this regard, reference to such objects as real even if not presently existing would seem hardly controversial—or at least not at that stage of conceptual development where our decentering abilities enable us to construct referential concepts that respect other points of view causally connected with our own.

Finally, it should be noted that whereas the original Diodorean notion of possibility results in the modal logic **S4.3**, i.e. the system **S4** plus the additional thesis

$$\Diamond^f \varphi \wedge \Diamond^f \psi \rightarrow \Diamond^f (\varphi \wedge \psi) \vee \Diamond^f (\varphi \wedge \Diamond^f \psi) \vee \Diamond^f (\psi \wedge \Diamond^f \varphi),$$

the same Diodorean notion of possibility, but redefined in terms of $\mathcal{F}_c$ instead, results in the modal logic **S4**. If we also assume, as is usual in special relativity, that the causal futures of any two moments t, t' of two local times of a causally connected system $\langle K, S \rangle$ *eventually* intersect, i.e. that there is a local time $\langle W, R, E \rangle \in K$ and a moment $w \in W$ such that tSw and $t'Sw$, then the thesis

$$\mathcal{F}_c \neg \mathcal{F}_c \neg \varphi \rightarrow \neg \mathcal{F}_c \neg \mathcal{F}_c \varphi$$

will be validated, and the Diodorean modality defined in terms of $\mathcal{F}_c$ will result in the modal system **S4.2** (cf. [Prior, 1967, p. 203]), i.e. the system **S4** plus the thesis

$$\Diamond^{fc} \Box^{fc} \varphi \rightarrow \Box^{fc} \Diamond^{fc} \varphi.$$

Many other modal concepts, it is clear, can also be characterised in terms of the semantics of a causally connected system of local times, including, e.g. the notion of something being necessary because of the way the past has been. What is distinctive about them all, moreover, is the unproblematic sense in which they can be taken as material or metaphysical modalities. This may indeed not be all there is to such a modality, but taking account of more will confront us once again with the problem of providing a philosophically coherent interpretation of the secondary semantics for such.

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TENSE AND TIME

1 INTRODUCTION

The semantics of tense has received a great deal of attention in the contemporary linguistics, philosophy, and logic literatures. This is probably due partly to a renewed appreciation for the fact that issues involving tense touch on certain issues of philosophical importance (viz., determinism, causality, and the nature of events, of time and of change). It may also be due partly to neglect. Tense was noticeably omitted from the theories of meaning advanced in previous generations. In the writings of both Russell and Frege there is the suggestion that tense would be absent altogether from an ideal or scientifically adequate language. Finally, in recent years there has been a greater recognition of the important role that all of the so-called indexical expressions must play in an explanation of mental states and human behavior. Tense is no exception. Knowing that one's friend *died* is cause for mourning, knowing that he *dies* is just another confirmation of a familiar syllogism.

This article will survey some attempts to make explicit the truth conditions of English tenses, with occasional discussion of other languages. We begin in Section 2 by discussing the most influential early scholarship on the semantics of tense, that of Jespersen, Reichenbach, and Montague. In Section 3 we outline the issues that have been central to the more linguistically-oriented work since Montague's time. Finally, in Section 4 we discuss recent developments in the area of tense logic, attempting to clarify their significance for the study of the truth-conditional semantics of tense in natural language.

2 EARLY WORK

2.1 *Jespersen*

The earliest comprehensive treatment of tense and aspect with direct influence on contemporary writings is that of Otto Jespersen. Jespersen's *A Modern English Grammar on Historical Principles* was published in seven volumes from 1909 to 1949. Jespersen's grammar includes much of what we would call semantics and (since he seems to accept some kind of identification between meaning and use) a good deal of pragmatics as well. The

aims and methods of Jespersen's semantic investigations, however, are not quite the same as ours.¹

First, Jespersen is more interested than we are in cataloging and systematizing the various uses of particular English constructions and less interested in trying to characterize their meanings in a precise way. This leads him to discuss seriously uses we would consider too obscure or idiomatic to bother with. For example, Jespersen notes in the *Grammar* that the expressions of the form *I have got A* and *I had got A* are different than other present perfect and past perfect sentences. *I have got a body*, for example, is true even though there was no past time at which an already existent me received a body. Jespersen suggests *I have in my possession* and *I had in my possession* as readings for *I have got* and *I had got*. And this discussion is considered important enough to be included in his *Essentials of English Grammar*, a one volume summary of the *Grammar*.

Jespersen however does *not* see his task as being merely to collect and classify rare flora. He criticizes Henry Sweet, for example, for a survey of English verb forms that includes such paradigms as *I have been being seen* and *I shall be being seen* on the grounds that they are so extremely rare that it is better to leave them out of account altogether. Nevertheless there is an *emphasis* on cataloging, and this emphasis is probably what leads Jespersen to adhere to a methodological principle that we would ignore; viz., that example sentences should be drawn from published literature wherever possible rather than manufactured by the grammarian. Contemporary linguists and philosophers of language see themselves as investigating fundamental intuitions shared by all members of a linguistic community. For this reason it is quite legitimate for them to produce a sentence and assert without evidence that it is well-formed or ill-formed, ambiguous or univocal, meaningful or unmeaningful. This practice has obvious dangers. Jespersen's methodological scruples, however, provide no real safety. On the one hand, if one limits one's examples to a small group of masters of the language one will leave out a great deal of commonly accepted usage. On the other hand, one can't accept *anything* as a legitimate part of the language just because it has appeared in print. Jespersen himself criticizes a contemporary by saying of his examples that 'these three passages are the only ones adduced from the entire English literature during nearly one thousand years'.

A final respect in which Jespersen differs from the other authors discussed here is his concern with the recent history of the language. Although the *Grammar* aims to be a compendium of contemporary idiom, the history of a construction is recited whenever Jespersen feels that such a discussion might be illuminating about present usage. A good proportion of the discussion of the progressive form, for example, is devoted to Jespersen's

¹By 'ours' we mean those of the authors discussed in the remainder of the article. Some recent work, like that of F. Palmer and R. Huddleston, is more in the tradition of Jespersen than this.

thesis that *I am reading* is a relatively recent corruption of *I am a-reading* or *I am on reading*, a construction that survives today in expressions like *I am asleep* and *I am ashore*. This observation, Jespersen feels, has enabled him to understand the meaning of the progressive better than his contemporaries.² In discussing Jespersen's treatment of tense and aspect, no attempt will be made to separate what is original with Jespersen from what is borrowed from other authors. Jespersen's grammar obviously extends a long tradition. See Binnick for a recent survey.³ Furthermore there is a long list of grammarians contemporaneous with Jespersen who independently produced analyses of tenses. See, for example, Curme, Kruisinga and Poutsma. Jespersen, however, is particularly thorough and insightful and, unlike his predecessors and contemporaries, he continues to be widely read (or at least cited) by linguists and philosophers. Jespersen's treatment of tense and aspect in English can be summarized as follows:

2.1.1 *Time*

It is important to distinguish *time* from *tense*. Tense is the linguistic device which is used (among other things) for expressing time relations. For example, *I start tomorrow* is a present tense statement about a future time. To avoid time-tense confusion it is better to reserve the term *past* for time and to use *preterit* and *pluperfect* for the linguistic forms that are more commonly called past tense and past perfect. Time must be thought of as something that can be represented by a straight line, divided by the present moment into two parts: the past and the future. Within each of the two divisions we may refer to some point as lying either before or after the main point of which we are speaking. For each of the seven resulting divisions of time there are *retrospective* and *prospective* versions. These two notions are not really a part of time itself, but have rather to do with the perspective from which an event on the time line is viewed. The prospective present time, for example, is a variety of present that looks forward into the future. In summary, time can be pictured as in Figure 2.1.1. The three divisions marked with *A*'s are past; those marked with *C*'s are future. The short pointed lines at each division indicate retrospective and prospective times.

2.1.2 *Tense morphology*

The English verb has only two tenses proper, the present tense and the preterit. There are also two tense phrases, the perfect (e.g., *I have written*) and the pluperfect or anteperfect (e.g., *I had written*). (Modal verbs,

²A similar claim is made in Vlach [1981]. For the most part, however, the history of English is ignored in contemporary semantics.

³Many of the older grammars have been reprinted in the series *English Linguistics: 1500–1800 (A Collection of Facsimile Reprints)* edited by R.C. Alston and published by Scholar Press Limited, Menston, England in 1967.

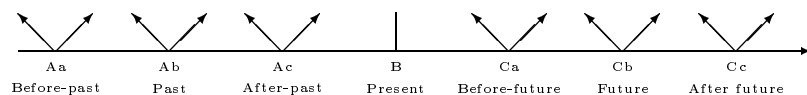


Figure 1.

including *can*, *may*, *must*, *ought*, *shall*, and *will*, cannot form perfects and pluperfects.) Corresponding to each of the four tenses and tense phrases there is an *expanded* (what is more commonly called today the *progressive*) form. For example, *had been writing* is the expanded pluperfect of *write*. It is customary to admit also future and future perfect tenses, as in *I will write* and *I shall have written*. But these constructions lack the fixity of the others. On the one hand, they are often used to express nontemporal ideas (e.g., volition, obstinacy) and on the other hand future time can be indicated in many other ways.

The present tense is primarily used about the present time, by which we mean an interval containing the present moment whose length varies according to circumstances. Thus the time we are talking about in *He is hungry* is shorter than in *None but the brave deserve the fair*. Tense tells us nothing about the duration of that time. The same use of present is found in expressions of intermittent occurrences (*I get up every morning at seven* and *Whenever he calls, he sits close to the fire*). Different uses of the present occur in statements of what might be found at all times by all readers (*Milton defends the liberty of the press in his Areopagitica*) and in expressions of feeling about what is just happening or has just happened (*That's capital!*). The present can also be used to refer to past times. For example, the *dramatic* or *historical* present can alternate with the preterit: *He perceived the surprise, and immediately pulls a bottle out of his pocket, and gave me a dram of cordial*. And the present can play the same role as the perfect in subordinate clauses beginning with *after*: *What happens to the sheep after they take its kidney out?* Present tense can be used to refer to future time when the action described is considered part of a plan already fixed: *I start for Italy on Monday*. The present tense can also refer to future events when it follows *I hope*, *as soon as*, *before*, or *until*.

The perfect is actually a kind of present tense that seems to connect the present time with the past. It is both a retrospective present, which looks upon the present as a result of what happened in the past and an inclusive present, which speaks of a state that is continued from the past into the present time (or at least one that has results or consequences bearing on the present time).

The preterit differs from the perfect in that it refers to some time in the past without telling anything about its connection with the present moment. Thus *Did you finish?* refers to a past time while *Have you finished?* is a question about present status. It follows that the preterit is appropriate with words like *yesterday* and *last year* while the perfect is better with *today*, *until now* and *already*. *This morning* requires a perfect tense when uttered in the morning and a preterit in the afternoon. Often the correct form is determined by context. For example, in discussing a schoolmate's Milton course, *Did you read Samson Agonistes?* is appropriate, whereas in a more general discussion *Have you read Samson Agonistes?* would be better. In comparing past conditions with present the preterit may be used (*English is not what it was*), but otherwise vague times are not expressed with the preterit but rather by means of the phrase *used to* (*I used to live at Chelsea*). The perfect often seems to imply repetition where the preterit would not. (Compare *When I have been in London*, with *When I was in London*).

The pluperfect serves primarily to denote before-past time or retrospective past, two things which cannot easily be kept apart. (An example of the latter use is *He had read the whole book before noon*.) After *after*, *when*, or *as soon as*, the pluperfect is interchangeable with the preterit.

The expanded tenses indicate that the action or state denoted provides a temporal frame encompassing something else described in the sentence or understood from context. For example, if we say *He was writing when I entered*, we mean that his writing (which may or may not be completed now) had begun, but was not completed, at the moment I entered. In the expanded present the shorter time framed by the expanded time is generally considered to be *very recently*. The expanded tenses also serve some other purposes. In narration simple tenses serve to carry a story forward while expanded tenses have a retarding effect. In other cases expanded tense forms may be used in place of the corresponding simple forms to indicate that a fact is already known rather than new, that an action is incomplete rather than complete or that an act is habitual rather than momentary. Finally, the expanded form is used in two clauses of a sentence to mark the simultaneity of the actions described. (In that case neither really frames the other.)

In addition to the uses already discussed, all the tenses can have somewhat different functions in passive sentences and in indirect speech. They also have uses apparently unrelated to temporal reference. For example, forms which are primarily used to indicate past time are often used to denote unreality, impossibility, improbability or non-fulfillment, as in *If John had arrived on time, he would have won the prize*.⁴

⁴From the contemporary perspective we would probably prefer to say here that *had*

2.1.3 Tense syntax

In the preceding discussion we started with the English tense forms and inquired about their meanings. Alternatively we can start with various temporal notions and ask how they can be expressed in English. If we do so, several additional facts emerge:

1. The future time can be denoted by present tense (*He **leaves** on Monday*), expanded present tense (*I **am dining** with him on Monday*), *is sure to*, *will*, *shall*, *come to* or *get to*.
2. The after-past can be expressed by *would*, *should*, *was to*, *was destined to*, expanded preterit (*They **were going out** that evening* and *When he came back from the club she **was dressing***) or *came to* (*In a few years he **came to control** all the activity of the great firm*).
3. The before-future can be expressed by *shall have*, *will have* or present (*I shall let you know as soon as I **hear** from them* or ***Wait** until the rain stops*).
4. The after-future is expressed by the same means as the future (*If you come at seven, dinner **will soon** be ready*).
5. Retrospective pasts and futures are not distinguished in English from before-pasts and before-futures. (But retrospective presents, as we have seen, are distinct from pasts. The former are expressed by the perfect, the latter by the preterit.)
6. Prospectives of the various times can be indicated by inserting expressions like *on the point of*, *about to* or *going to*. For example, *She is about to cry* is a prospective present.

2.2 Reichenbach

In his general outlook Reichenbach makes a sharp and deliberate break with the tradition of grammarians like Jespersen. Jespersen saw himself as studying the English language by any means that might prove useful (including historical and comparative investigations). Reichenbach saw himself as applying the methods of contemporary logic in a new arena. Thus, while Jespersen's writings about English comprise a half dozen scholarly treatises, Reichenbach's are contained in a chapter of an introductory logic text. (His treatment of tense occupies twelve pages.) Where Jespersen catalogs dozens of uses for an English construction, Reichenbach is content to try to characterize carefully a single use and then to point out that this paradigm does not cover all the cases. While Jespersen uses, and occasionally praises, the *arrived* is a subjunctive preterit which happens to have the same form as a pluperfect.

efforts of antecedent and contemporary grammarians, Reichenbach declares that the state of traditional grammar is hopelessly muddled by its two-millennial ties to a logic that cannot account even for the simplest linguistic forms.

Despite this difference in general outlook, however, the treatment of tenses in Reichenbach is quite similar to that in Jespersen. Reichenbach's chief contribution was probably to recognize the importance of the distinction between what he calls the *point of the event* and the *point of reference* (and the relative unimportance and obscurity of Jespersen's notions of prospective and retrospective time.) In the sentence *Peter had gone*, according to Reichenbach, the point of the event is the time when Peter went. The point of reference is a time between this point and the point of speech, whose exact location must be determined by context. Thus Reichenbach's account of the past perfect is very similar to Jespersen's explanation that the past perfect indicates a 'before past' time. Reichenbach goes beyond Jespersen, however, in two ways.

First, Reichenbach is a little more explicit about his notion of reference times than is Jespersen about the time of which we are speaking. He identifies the reference time in a series of examples and mentions several rules that might be useful in determining the reference time in other examples. Temporally specific adverbials like *yesterday*, *now* or *November 7, 1944*, for example, are said to refer to the reference point. Similarly, words like *when*, *after*, and *before* relate the reference time of a adjunct clause to that of the main clause. And if a sentence does not say anything about the relations among the reference times of its clauses, then every clause has the same point of reference.

Second, Reichenbach argues that the notion of reference time plays an important role in *all* the tenses. The present perfect, for example, is distinguished by the fact that the event point is before the point of reference and the point of reference coincides with the point of speech. (So *I have seen Sharon* has the same meaning as *Now I have seen Sharon*.) In general, each tense is determined by the relative order of the point of event (*E*), the point of speech (*S*), and the point of reference (*R*). If *R* precedes *S* we have a kind of past tense, if *S* precedes *R* we have a kind of future tense and if *R* coincides with *S* we have a kind of present. This explains Jespersen's feeling that the simple perfect is a variety of the present. Similarly the labels 'anterior', 'posterior' and 'simple' indicate that *E* precedes, succeeds or coincides with *R*. The account is summarized in the following table.

Each of the tenses on this table also has an expanded form which indicates, according to Reichenbach, that the event covers a certain stretch of time.

Notice that the list of possible tenses is beginning to resemble more closely the list of tenses realized in English. According to Jespersen there are seven divisions of time, each with simple, retrospective and prospective versions. This makes twenty-one possible tenses. According to Reichenbach's scheme

Structure	New Name	Traditional Name
E_R_S	Anterior past	Past perfect
E, R_S	Simple past	Simple past
R_E_S		
R_S, E	Posterior past	_____
R_S_E		
E_S, R	Anterior present	Present perfect
S, R, E	Simple present	Present
S, R_E	Posterior present	Simple future
S_E_R		
S, E_R	Anterior future	Future perfect
E_S_R		
S_R, E	Simple future	Simple future
S_R_E	Posterior future	_____

there should be thirteen possible tenses, corresponding to the thirteen orderings of E , S , and R . Looking more closely at Reichenbach, however, we see that the *tense* of a sentence is determined only by the relative order of S and R , and the *aspect* by the relative order of R and E . Since there are three possible orderings of S and R , and independently three possible orderings of R and E , there are really only nine possible complex tenses (seven of which are actually realized in English).⁵

Finally, Reichenbach acknowledges that actual language does not always keep to the scheme set forth. The expanded forms, for example, sometimes indicate repetition rather than duration: *Women are wearing larger hats this year*. And the present perfect is used to indicate that the event has a certain duration which reaches up to the point of speech: *I have lived here for ten years*.

2.3 Montague

Despite Reichenbach's rhetoric, it is probably Montague, rather than Reichenbach, who should be credited with showing that modern logic can be fruitfully applied to the study of natural language. Montague actually had very little to say about tense, but his writings on language have been very influential among those who do have something to say. Two general principles underlie Montague's approach.

⁵There are actually only six English tense constructions on Reichenbach's count, because two tenses are realized by one construction. The simple future is ambiguous between S, R_E , as in *Now I shall go* or S_R, E , as in *I shall go tomorrow*. Reichenbach suggests that, in French the two tenses may be expressed by different constructions: *je vais voir* and *je verrai*.

- (1a) Compositionality. The meaning of an expression is determined by the meaning of its parts.
- (1b) Truth conditions. The meaning of a declarative sentence is something that determines the conditions under which that sentence is true.

Neither of these principles, of course, is original with Montague, but it is Montague who shows how these principles can be used to motivate an explicit account of the semantics of particular English expressions.

Initially, logic served only as a kind of paradigm for how this can be done. One starts with precisely delineated sets of *basic expressions* of various categories. *Syntactic rules* show how complex expressions can be generated from the basic ones. A class of permissible *models* is specified, each of which assigns interpretations to the basic expressions. *Rules of interpretation* show how the interpretation of complex expressions can be calculated from the interpretations of the expressions from which they are built.

The language of classical predicate logic, for example, contains predicates, individual variables, quantifiers, sentential connectives, and perhaps function symbols. Generalizations of this logic are obtained by adding additional expressions of these categories (as is done in modal and tense logic) or by adding additional categories (as is done in higher order logics). It was Montague's contention that if one generalized enough, one could eventually get English itself. Moreover, clues to the *direction* this generalization should take are provided by modal and tense logic. Here sentences are interpreted by functions from possible worlds (or times or *indices* representing aspects of context) to truth values. English, for Montague, is merely an exceedingly baroque intensional logic. To make this hypothesis plausible, Montague constructed, in [1970; 1970a] and [1973], three 'fragments' of English of increasing complexity. In his final fragment, commonly referred to as PTQ, Montague finds it convenient to show how the expressions can be translated into an already-interpreted intensional logic rather than to specify an interpretation directly. The goal is now to find a translation procedure by which every expression of English can be translated into a (comparatively simple) intensional logic.

We will not attempt here to present a general summary of PTQ. (Readable introductions to Montague's ideas can be found in Montague [1974] and Dowty [1981].) We will, however, try to describe its treatment of tense. To do so requires a little notation. Montague's intensional logic contains tense operators **W** and **H** meaning roughly *it will be the case that* and *it was the case that*. It also contains an operator $\wedge$ that makes it possible to refer to the intension of an expression. For example, if **a** is an expression referring to the object **a**, then $\wedge\mathbf{a}$ denotes the function that assigns **a** to every pair of a possible world **w** and a time **t**.

Among the expressions of English are *terms* and *intransitive verb phrases*. An intransitive verb phrase B is translated by an expression $\mathbf{B}'$ which denotes a function from entities to truth values. (That is, $\mathbf{B}'$ is of type $\langle e, t \rangle$.) A term A is translated by an expression $\mathbf{A}'$ which denotes a function whose domain is intensions of functions from entities to truth values and whose range is truth values. (That is, $\mathbf{A}'$ is of type $\langle \langle s, \langle e, t \rangle, t \rangle \rangle$.) Tense and negation in PTQ are treated together. There are six ways in which a term may be combined with an intransitive verb phrase to form a sentence. These generate sentences in the present, future, present perfect, negated present, negated future and negated present perfect forms. The rules of translation corresponding to these six constructions are quite simple. If B is an intransitive verb phrase with translation $\mathbf{B}'$ and A is a term with translation $\mathbf{A}'$ then the translations of the six kinds of sentences that can be formed by combining A and B are just $\mathbf{A}'(\wedge \mathbf{B}')$, $\mathbf{WA}'(\wedge \mathbf{B}')$, $\mathbf{HA}'(\wedge \mathbf{B}')$, $\neg \mathbf{A}'(\wedge \mathbf{B}')$, $\neg \mathbf{WA}'(\wedge \mathbf{B}')$ and $\neg \mathbf{HA}'(\wedge \mathbf{B}')$.

A simple example will illustrate. Suppose that A is *Mary* and that B is *sleeps*. The future tense sentence *Mary will sleep* is assigned translation $\mathbf{WMary}(\wedge \mathbf{sleeps})$. $\mathbf{Mary}$ denotes that function which assigns 'true' to a property P in world w at time t if and only if Mary has P in w at t . The expression $\wedge \mathbf{sleeps}$ denotes the property of sleeping, i.e. the function f from intensions to functions from individuals to truth values such that $f(\langle w, t \rangle)(a) = \text{'true'}$ if and only if a is an individual who is asleep in world w at time t (for any world w , time t , and individual a). Thus $\mathbf{Mary}(\wedge \mathbf{sleeps})$ will be true at $\langle w, t \rangle$ if and only if Mary is asleep in w at t . Finally, the sentence $\mathbf{WMary}(\wedge \mathbf{sleeps})$ is true in a world w at a time t if and only if $\mathbf{Mary}(\wedge \mathbf{sleeps})$ is true at some $\langle w, t' \rangle$, where t' is a later time than t .

This treatment is obviously crude and incomplete. It was probably intended merely as an illustration of how tense *might* be handled within Montague's framework. Nevertheless, it contains the interesting observation that the past tense operator found in the usual tense logics corresponds more closely to the present perfect tense than it does to the past. In saying *John has kissed Mary* we seem to be saying that there was *some* time in the past when *John kisses Mary* was true. In saying *John kissed Mary*, we seem to be saying that *John kisses Mary* was true at *the* time we happen to be talking about. This distinction between *definite* and *indefinite* past times was pointed out by Jespersen, but Jespersen does not seem to have thought it relevant to the distinction between present perfect and past.

Reichenbach's use of both event time and reference time, leading to a three-dimensional logic, may suggest that it will not be easy to add the past tenses to a PTQ-like framework. However, one of the differences between Reichenbach's reference time and event time seems to be that the former is often fixed by an adverbial clause or by contextual information whereas the latter is less often so fixed. So it is approximately correct to say that the reference time is determinate whereas the event time is indetermi-

nate. This may help explain the frequent remarks that only two times are needed to specify the truth conditions of all the tenses. In one sense these remarks are wrong. S , R and E all play essential roles in Reichenbach's explanation of the tenses. But only S and R ever need to be extracted from the context. All that we need to know about E is its position relative to R and this information is contained in the sentence itself. Thus a tense logic following Reichenbach's analysis could be two-dimensional, rather than three-dimensional. If s and r are the points of speech and reference, for example, we would have $(s, r) \models \text{PASTPERFECT}(A)$ if and only if $r < s$ and, for some $t < r$, $t \models A$. (See Section 4 below.)

Still, it seems clear that the past tenses cannot be added to PTQ without adding something like Reichenbach's point of reference to the models. Moreover, adherence the idea that there should be a separate way of combining tenses and intransitive verb phrases for every negated and unnegated tense would be cumbersome and would miss important generalizations. Montague's most important legacies to the study of tense were probably his identification of meaning with truth conditions, and his high standards of rigor and precision. It is striking that Jespersen, Reichenbach and Montague say successively less about tense with correspondingly greater precision. A great deal of the contemporary work on the subject can be seen as an attempt to recapture the insights of Jespersen without sacrificing Montague's precision.

3 CONTEMPORARY VIEWS

In Sections 3.1 and 3.2 below we outline what seem to us to be two key issues underlying contemporary research into the semantics of tense. The first has to do with whether tense should be analyzed as an operator or as something that refers to particular time or times; this is essentially a type-theoretic issue. The second pertains to a pair of truth-conditional questions which apparently are often confused with the type-theoretic ones: (i) *does the semantics of tense involve quantification over times, and if so how does this quantification arise?*, and (ii) *to what extent is the set of times relevant to a particular tensed sentence restricted or made determinate by linguistic or contextual factors?* Section 3.3 then outlines how contemporary analytical frameworks have answered these questions. Finally, Section 3.4 examines in more detail some of the proposals which have been made within these frameworks about the interpretation of particular tenses and aspects.

3.1 Types for Tense

The analyses of Reichenbach and Montague have served as inspiration for two groups of theorists. Montague's approach is the one more familiar

from traditional tense logics developed by Prior and others. The simplest non-syncategorematic treatment of tense which could be seen as essentially that of Montague would make tenses propositional operators, expressions of type $\langle\langle s, t \rangle, t\rangle$ or $\langle\langle s, t \rangle, \langle s, t \rangle\rangle$, that is, either as functions from propositions to truth values or as functions from propositions to propositions (where propositions are taken to be sets of world-time pairs). For example, the present perfect might have the following interpretation:

- (2) **PrP** denotes that function f from propositions to propositions such that, for any proposition p , $f(p)$ = the proposition q , where for any world w and time t , $q(\langle w, t \rangle)$ = ‘true’ iff for some time t' preceding t , $p(\langle w, t' \rangle)$ = ‘true’.

Two alternative, but closely related, views would take tense to have the type of a verb phrase modifier $\langle\langle s, \langle e, t \rangle \rangle, \langle e, t \rangle\rangle$ ([Bäuerle, 1979; Kuhn, 1983]) or as a ‘mode of combination’ in $\langle\text{type}(\text{TERM}), \langle\langle s, \langle e, t \rangle \rangle, t \rangle\rangle$ or $\langle\langle s, \langle e, t \rangle \rangle, \langle\text{type}(\text{TERM}), t \rangle\rangle$. We will refer to these approaches as representative of the *operator* view of tense.

The alternative approach is more directly inspired by Reichenbach’s views. It takes the semantics of tense to involve reference to particular times. This approach is most thoroughly worked out within the framework of Discourse Representation Theory (DRT; [Kamp, 1983; Kamp and Rohrer, 1983; Hinrichs, 1986; Partee, 1984]), but for clarity we will consider the type-theoretic commitments of the neo-Reichenbachian point of view through the use of a Predicate Calculus-like notation. We may take a tense morpheme to introduce a free variable to which a time can be assigned. Depending on which tense morpheme is involved, the permissible values of the variable should be constrained to fall within an appropriate interval. For example, the sentence *Mary slept* might have a logical form as in (3).

- (3) **PAST**(t) & **AT**(t , **sleeps**(**Mary**)).

With respect to an assignment g of values to variables, (3) should be true if and only if $g(t)$ is a time that precedes the utterance time and one at which Mary sleeps. On this approach the semantics of tense is analogous to that of pronouns, a contention defended most persuasively by Partee.

A more obviously Reichenbachian version of this kind of analysis would introduce more free variables than simply t in (3). For example, the pluperfect *Mary had slept* might be rendered as in (4):

- (4) **PAST**(r) & $t < r$ & **AT**(t , **sleeps**(**Mary**)).

This general point of view could be spelled out in a wide variety of ways. For example, times might be taken as arguments of predicates, or events and states might replace times. We refer to this family of views as *referential*.

3.2 Quantification and determinacy

3.2.1 Quantification

In general, the operator theory has taken tense to involve quantification over times. Quantification is not an inherent part of the approach, however; one might propose a semantics for the past tense of the following sort:

- (5) $(r, u) \models \mathbf{PAST}(S)$ iff $r < u$ and $(r, r) \models S$.

Such an analysis of a non-quantificational past tense might be seen as especially attractive if there are other tense forms that are essentially quantificational. An operator-based semantics would be a natural way to introduce this quantification, and in the interest of consistency one might then prefer to treat all tenses as operators—just as PTQ argues that all NP's are quantifiers because some are inherently quantificational. On the other hand, if no tenses are actually quantificational it might be preferable to utilize a less powerful overall framework.

The issue of quantification for the referential theory of tense is not entirely clear either. If there are sentences whose truth conditions must be described in terms of quantification over times, the referential theory cannot attribute such quantification to the tense morpheme. But this does not mean that such facts are necessarily incompatible with the referential view. Quantification over times may arise through a variety of other, more general, means. Within DRT and related frameworks, several possibilities have been discussed. The first is that some other element in the sentence may bind the temporal variable introduced by tense. An adverb of quantification like *always*, *usually*, or *never* would be the classical candidate for this role.

- (6) When it rained, it always poured.

- (7) $\forall t[(\mathbf{PAST}(t) \ \& \ \mathbf{AT}(t, \mathbf{it-rains})) \rightarrow (\mathbf{PAST}(t) \ \& \ \mathbf{AT}(t, \mathbf{it-pours}))]$.

DRT follows Lewis [1975] in proposing that *always* is an unselective universal quantifier which may bind any variables present in the sentence. Hinrichs and Partee point out that in some cases it may turn out that a variable introduced by tense is thus bound; their proposals amount to assigning (6) a semantic analysis along the lines of (7).

The other way in which quantification over times may arise in referential analyses of tense is through some form of default process. The most straightforward view along these lines proposes that, in the absence of explicit quantificational adverbs, the free variable present in a translation like (3), repeated here, is subject to a special rule that turns it into a quantified formula like (8):

(3) **PAST**(t) & **AT**(t , **sleeps**(**Mary**)).

(8) $\exists t$ [**PAST**(t) & **AT**(t , **sleeps**(**Mary**))].

This operation is referred to as *existential closure* by Heim; something similar is proposed by Parsons [1995]. It is also possible to get the effect of existential quantification over times through the way in which the truth of a formula is defined. This approach is taken by DRT as well as Heim [1982, Ch. III]. For example, a formula like (3) would be true with respect to a model M if and only if there is *some* function g from free variables in (3) to appropriate referents in M such that $g(t)$ precedes the utterance time in M and $g(t)$ is a time at which Mary is asleep in M .

To summarize, we may say that one motivation for the operator theory of tense comes from the view that some tense morphemes are inherently quantificational. The referential analysis, in contrast, argues that all examples of temporal quantification are to be attributed not to tense but to independently needed processes.

3.2.2 *Determinacy*

An issue which is often not clearly distinguished from questions of the type and quantificational status of tense is that of the determinacy or definiteness of tense. Classical operator-based tense logics treat tense as all but completely indeterminate: a past tense sentence is true if and only if the untensed version is true at *any* past time. On the other hand, Reichenbach's referential theory seemingly considers tense to be completely determinate: a sentence is true or false with respect to the *particular* utterance time, reference time, and event time appropriate for it. However, we have already seen that a referential theory might allow that a time variable can be bound by some quantificational element, thus rendering the temporal reference less determinate. Likewise, we have seen that an operator-based theory may be compatible with completely determinate temporal reference, as in (5). In this section, we would like to point out how varying degrees of determinacy can be captured within the two systems.

If temporal reference is fully indeterminate, it is natural to adopt an operator view: **PAST**(B) is true at t if and only if B is true at some $t' < t$. A referential theory must propose that in every case the time variable introduced by tense is bound by some quantificational operator (or effectively quantified over by default, perhaps merely through the effects of the truth definition). In such cases it seems inappropriate to view the temporal parameters as 'referring' to times.

If temporal reference is fully determinate, the referential theory need make no appeal to any ancillary quantification devices. The operator theory may use a semantics along the lines of (3). Alternatively, tense might be seen as an ordinary quantificational operator whose domain of quantification has

been severely restricted. We might implement this idea as follows: Suppose that each tense morpheme bears an index, as *Mary PAST₃sleeps*. Sentences are interpreted with respect to a function R from indices to intervals. (The precedence order is extended from instants to intervals and instants in the appropriate way, with $<$ indicating ‘completely precedes’.) The formula in (9a) would then have the truth conditions of (9b).

(9a) **PAST₃ (sleeps(Mary))**.

(9b) $(R, u) \models \mathbf{PAST}_3(\mathbf{sleeps(Mary)})$ iff for some time $t \in R(3)$, $t < u$ and $(R, t) \models \mathbf{sleeps(Mary)}$.

Plainly, R in (9b) is providing something very similar to that of the reference time in Reichenbach’s system. This can be seen by the fact that the identity of $R(3)$ should be fixed by temporal adverbs like *yesterday*, as in *Yesterday, Mary slept*.

Finally, we should examine what could be said about instances of tense which are partially determinate. The immediately preceding discussion makes it clear what the status of such examples would be within an operator account; they would simply exemplify restricted quantification ([Bennett and Partee, 1972; Kuhn, 1979]). Instead of the analysis in (9), we would propose that R is a function from indices to sets of intervals, and give the truth conditions as in (10).

(10) $(R, u) \models \mathbf{PAST}_3(\mathbf{sleeps(Mary)})$ iff for some time $t \in R(3)$, $t < u$ and $(R, t) \models \mathbf{sleeps(Mary)}$.

According to (10), (9a) is true if and only if Mary was asleep at some past time which is within the set of contextually relevant past times. Temporal quantification would thus be seen as no different from ordinary nominal quantification, as when *Everyone came to the party* is taken to assert that everyone *relevant* came to the party.

Referential analyses of tense would have to propose that partial determinacy arises when temporal variables are bound by restricted quantifiers. Let us consider a Reichenbach-style account of *Mary slept* along the lines of (11).

(11) $\exists t [\mathbf{PAST}(r) \ \& \ t \in r \ \& \ \mathbf{AT}(t, \mathbf{sleeps(Mary)})]$.

The remaining free variable in (11), namely r , will have to get its value (the reference *set*) from the assignment function g . The formula in (11) has $t \in r$ where Reichenbach would have $t = r$; the latter would result in completely determinate semantics for tense, while (11) results in restricted quantification. The sentence is true if and only if Mary slept during some past interval contained in $g(r)$.

The only difference between (10) and (11) is whether the quantificational restriction is represented in the translation language as a variable, the r in (11), or as a special index on the operator, the subscripted $\exists$ in (10). In each case, one parameter of interpretation must be some function which identifies the set of relevant times for the quantification. In (11), it is the assignment function, g , while in (10) it is R . Clearly at this point the differences between the two theories are minor. To summarize, we need to distinguish three closely related ways in which theories of tense may differ: (i) They may take tense to be an operator or to introduce elements which refer to times; (ii) they may involve quantification over times through a considerable variety of means—the inherent semantics of tense itself, the presence of some other quantificational element within the sentence, or a default rule; and (iii) they may postulate that the temporal reference of sentences is fully determinate, fully indeterminate, or only partially determinate.

3.3 *Major contemporary frameworks*

Most contemporary formal work on the semantics of tense takes place within two frameworks: Interval Semantics and Discourse Representation Theory. In this section we describe the basic commitments of each of these, noting in particular how they settle the issues discussed in 3.1 and 3.2 above. We will then consider in a similar vein a couple of other influential viewpoints, those of Situation Semantics [Cooper, 1986] and the work of Enç [1986; 1987].

By *Interval Semantics* we refer to the framework which has developed out of the Intensional Logic of Montague's PTQ. There are a number of implementations of a central set of ideas; for the most part these differ in fairly minor ways, such as whether quantification over times is to be accomplished via operators or explicit quantifiers. The key aspects of Interval Semantics are: (i) the temporal part of the model consists of set I of *intervals*, the set of open and closed intervals of the reals, with precedence and temporal overlap relations defined straightforwardly; (ii) the interpretation of sentences depends on an *evaluation interval* or *event time*, an *utterance time*, and perhaps a *reference interval* or set of reference intervals; (iii) interpretation proceeds by translating natural language sentences into some appropriate higher-order logic, typically an intensional λ -calculus; and (iv) tenses are translated by quantificational operators or formulas involving first-order quantification to the same effect. The motivation for (i) comes initially from the semantics for the progressive, a point which we will see in Section 3.4 below. We have already examined the motivation for (ii), though in what follows we will see more clearly what issues arise in trying to understand the relationship between the reference interval and the evaluation interval. Points (iii) and (iv) are implementation details with which we will not much concern ourselves.

From the preceding, it can be seen what claims Interval Semantics makes

concerning the issues in 3.1 and 3.2. Tense has the type of an operator. It is uniformly quantificational, but shows variable determinacy, depending on the nature of the reference interval or intervals.

Discourse Representation Theory is one of a number of theories of *dynamic interpretation* to be put forth since the early 1980's; others include File Change Semantics [Heim, 1982] and Dynamic Montague Grammar [Groenendijk and Stokhof, 1990]. What the dynamic theories share is a concern with the interpretation of multi-sentence texts, concentrating on establishing means by which information can be passed from one sentence to another. The original problems for which these theories were designed had to do with nominal anaphora, in particular the relationships between antecedents and pronouns in independent sentences like (12) and donkey sentences like (13).

(12) A man walked in. He sat down.

(13) When a man walks in, he always sits down.

Of the dynamic theories, by far the most work on tense has taken place within DRT. It will be important over time to determine whether the strengths and weaknesses of DRT analyses of tense carry over to the other dynamic approaches.

As noted above, work on tense within DRT has attempted to analogize the treatment of tense to that of nominal anaphora. This has resulted in an analytical framework with the following general features: (i) the temporal part of the model consists of a set of *eventualities* (events, processes, states, etc.), and possibly of a set of intervals as well; (ii) the semantic representation of a discourse (or sub-part thereof) contains explicit variables ranging over to reference times, events, and the utterance time; (iii) interpretation proceeds by building up a *Discourse Representation Structure* (DRS), a partial model consisting of a set of objects (*discourse markers*) and a set of conditions specifying properties of and relations among them; the discourse is true with respect to a model M if and only if the partial model (DRS) can be embedded in the full model M ; (iv) tenses are translated as conditions on discourse markers representing events and/or times. For example, consider the discourse in (14).

(14) Pedro entered the kitchen. He took off his coat.

We might end up with discourse markers representing Pedro (x), the kitchen (y), the coat (z), the event of entering the kitchen (e_1), the event of taking off the coat (e_2), the utterance time (u), the reference time for the first sentence (r_1) and the reference time for the second sentence (r_2). The DRS would contain at least the following conditions: **Pedro= x , kitchen(y), coat(z), entering(e_1, x, y), taking-off(e_2, x, z), $r_1 < u$, $r_2 < u$, $r_1 < r_2$, $e_1 \circ$**

r_1 , and $e_2 \circ r_2$ (where $\circ$ represents temporal overlap). The algorithms for introducing conditions may be rather complex, and typically are sensitive to the aspectual class of the eventualities represented (that is, whether they are events, processes, states, etc.).

DRT holds a referential theory of tense, treating it via discourse markers plus appropriate conditions. It therefore maintains that tense is not inherently quantificational, and that any quantificational force which is observed must come from either an independent operator, as with (6), or default rule. Given the definition of truth mentioned above, tense will be given a default existential quantificational force—the DRS for (14) will be true if there is *some* mapping from discourse markers to entities in the model satisfying the conditions. The DRT analysis of tense also implies that temporal reference is highly determinate, since the events described by a discourse typically must overlap temporally with a contextually determined reference time.

Closely related to the DRT view of tense are a pair of *indexical* theories of tense. The first is developed by Cooper within the framework of *Situation Semantics* (Barwise and Perry [1983]). Situation Semantics constructs objects known as *situations* or *states of affairs* set-theoretically out of properties, relations, and individuals (including space-time locations). Let us say that the situation of John loving Mary is represented as $\langle l, \langle \langle \text{love, John, Mary} \rangle, 1 \rangle \rangle$, l being a spatiotemporal location and 1 representing ‘truth’. A set of states of affairs is referred to as a *history*, and it is the function of a sentence to describe a history. A simple example is given in (15).

- (15) *John loved Mary* describes a history h with respect to a spatiotemporal location l iff $\langle l, \langle \langle \text{love, John, Mary} \rangle, 1 \rangle \rangle \in h$.

Unless some theory is given to explain how the location l is arrived at, a semantics like (15) will of course not enlighten us much as to the nature of tense. Cooper proposes that the location is provided by a *connections function*; for our purposes a connections function can be identified with a function from words to individuals. When the word is a verb, a connections function c will assign it a spatiotemporal location. Thus,

- (16) *John loved Mary* describes a history h with respect to a connections function c iff $\langle c(\text{loved}), \langle \langle \text{love, John, Mary} \rangle, 1 \rangle \rangle \in h$.

Cooper’s theory is properly described as an ‘indexical’ approach to tense, since a tensed verb directly picks out the location which the sentence is taken to describe.⁶

⁶Unlike ordinary indexicals, verbs do not refer to the locations which they pick out. The verb *loved* still denotes the relation **love**.

Enç's analysis of tense is somewhat similar to Cooper's. She proposes that tense morphemes refer to intervals. For example, the past tense morpheme *-ed* might refer, at an utterance time u , to the set of moments preceding u . For Enç, a verb is a semi-indexical expression, denoting a contextually relevant subrelation of the relation which it is normally taken to express—e.g., any occurrence of *kiss* will denote a subset of $\{\langle x, y \rangle: x \text{ kisses } y \text{ (at some time)}\}$. Tense serves as one way of determining which subrelation is denoted. The referent of a verb's tense morpheme serves to constrain the denotation of the verb, so that, for instance, the verb *kissed* must denote a set of pairs of individuals where the first kissed the second during the past, i.e. during the interval denoted by the tense.

- (17) *kissed* denotes a (contextually relevant) subset of $\{\langle x, y \rangle: \text{for some } t \in \text{-ed}, x \text{ kissed } y \text{ at } t\}$.

In (17), *-ed* is the set of times denoted by *-ed*, i.e. that set of times preceding the utterance time.

Both Enç's theory and the Situation Semantics approach outlined above seem to make the same commitments on the issues raised in Sections 3.1 and 3.2 as DRT. Both consider tense to be non-quantificational and highly determinate. They are clearly referential theories of tense, taking its function to be to pick out a particular time with respect to which the eventualities described by the sentence are temporally located.

3.4 *The compositional semantics of individual tenses and aspects*

Now that we have gone through a general outline of several frameworks which have been used to semantically analyze tense in natural language, we turn to seeing what specific claims have been made about the major tenses (present, past, and future) and aspects (progressive and perfect) in English.

3.4.1 *Tense*

Present Tense. In many contemporary accounts the semantic analysis of the present underlies that of all the other tenses.⁷ But despite this allegedly fundamental role, the only use of the present that seems to have been treated formally is the 'reportive' use, in which the sentence describes an event that is occurring or a state that obtains at the moment of utterance.⁸ The preoccupation with reportive sentences is unfortunate for two reasons. First, the reportive uses are often the less natural ones—consider the sentence

⁷This is true, for example, of Bennett and Partee. But there is no consensus here. Kuhn [1983], for example, argues that past, present, and future should be taken as (equally fundamental) modes of combination of noun phrases and verb phrases.

⁸Many authors restrict the use of the term 'reportive' to event sentences.

Jill walks to work (though many languages do not share this feature with English). Second, if the present tense is taken as fundamental, the omission of a reading in the present tense can be transferred to the other tenses. (*John walked to work* can mean that John habitually walked to work.) The neglect is understandable, however, in view of the variety of uses the present can have and the difficulty of analyzing them. One encounters immediately, for example, the issue discussed below.

Statives and non-statives. There is discussion in the philosophical literature beginning with Aristotle about the kinds of verb phrases there are and the kinds of things verb phrases can describe. Details of the classification and terminology vary widely. One reads about events, processes, accomplishments, achievements, states, activities and performances. The labels are sometimes applied to verb phrases, sometimes to sentences and sometimes to eventualities. There seems to be general agreement, however, that some kind of classification of this kind will be needed in a full account of the semantics of tense. In connection with the present tense there is a distinction between verb phrases for which the reportive sense is easy (e.g., *John knows Mary*, *The cat is on the mat*, *Sally is writing a book*) and those for which the reportive sense is difficult (e.g., *John swims in the channel*, *Mary writes a book*). This division almost coincides with a division between verb phrases that have a progressive form and those that do not. (Exceptions—noted by Bennett and Partee—include *John lives in Rome* and *John resides in Rome*, both of which have easy reportive uses but common progressive forms.) It also corresponds closely to a division of sentences according to the kind of *when* clauses they form. The sentence *John went to bed when the cat came in* indicates that John went to bed after the cat came in, while *John went to bed when the cat was on the mat* suggests that the cat remained on the mat for some time after John went to bed. In general, if the result of prefixing a sentence by *when* can be paraphrased using *just after* it will have difficult reportive uses and common progressive forms. If it can be paraphrased using *still at the time* it will have easy reportive uses and no common progressive forms. (Possible exceptions are ‘inceptive readings’ like *She smiled when she knew the answer*; see the discussion in Section 3.4.4 below.)

The correspondence among these three tests suggests that they reflect some fundamental ways in which language users divide the world. The usual suggestion is that sentences in the second class (easy reportive readings, no progressives and *when* = *still at the time*) describe *states*. States are distinguished by the fact that they seem to have no temporal parts. The way Emmon Bach puts it is that it is possible to imagine various states obtaining even in a world with only one time, whereas it is impossible to imagine events or processes in such a world. (Other properties that have been regarded as

characteristic of states are described in Section 4.2 below.) Sentences that describe states are *statives*; those that do not are *non-statives*.

There is some disagreement about whether sentences in the progressive are statives. The fact that Harry is building a house, for example, can go on at discontinuous intervals and the fact that Mary is swimming in the Channel is composed of a sequence of motions, none of which is itself swimming, lead Gabbay and Moravcsik to the conclusion that present progressives do not denote states. But according to the linguistic tests discussed above progressives clearly do belong with the state sentences. For this reason, Vlach, Bach, and Bennett all take the other side. The exact importance of this question depends on what status one assigns to the property of being a stative sentence. If it means that the sentence implies that a certain kind of eventuality known as a state obtains, then it seems that language users assume or pretend that there is some state that obtains steadily while Mary makes the swimming motions and another while Harry is involved in those house-building activities. On the other hand, if ‘stative’ is merely a label for a sentence with certain temporal properties, for example passing the tests mentioned above, then the challenge is just to assign a semantics to the progressive which gives progressive sentences the same properties as primitive statives; this alternative does not commit us to the actual existence of states (cf. Dowty’s work). Thus, the implications of deciding whether to treat progressives as statives depends on one’s overall analytical framework, in particular on the basic eventuality/time ontology one assumes.

A recent analysis of the present tense which relates to these issues has been put forth by Cooper. As mentioned above, Cooper works within the Situation Semantics framework, and is thereby committed to an analysis of tense as an element which describes a spatiotemporal region. A region of this kind is somewhat more like an eventuality, e.g. a state, than a mere interval of time; however, his analysis does not entail a full-blown eventuality theory in that it doesn’t (necessarily) propose primitive classes of states, events, processes, etc. Indeed, Cooper proposes to define states, activities, and accomplishments in terms very similar to those usual in interval semantics. For instance, stative and process sentences share the property of describing some temporally included sublocation of any spatiotemporal location which they describe (*temporal ill-foundedness*); this is a feature similar to the *subinterval property*, which arises in purely temporal analyses of the progressive (see 3.4.2 below).

Cooper argues that this kind of framework allows an explanation for the differing effects of using the simple present with stative, activity, and accomplishment sentences. The basic proposal about the present tense is that it describes a present spatiotemporal location—i.e. the location of discourse. Stative sentences have both temporal ill-foundedness and the property of *independence of space*, which states that, if they describe a location l , they also describe the location $l+$ which is l expanded to include

all of space. This means that if, for example, John loves Mary anywhere for a length of time including the utterance time, *John loves Mary* will describe all of space for the utterance time. This, according to Cooper, allows the easy use of the present tense here. It seems, though, that to get the result we need at least one more premise: either a stative must describe any spatial sublocation of any location it describes (so that it will precisely describe the utterance location) or we must count the location of utterance for a stative to include all of space.

Activity sentences do not have independence of space. This means that, if they are to be true in the present tense, the utterance location will have to correspond spatially to the event's location. This accounts for the immediacy of sentences like *Mary walks away*. On the other hand, they do have temporal ill-foundedness, which means that the sentence can be said even while the event is still going on. Finally, accomplishment sentences lack the two above properties but have *temporal well-foundedness*, a property requiring them not to describe of any temporal subpart of any location they describe. This means that the discourse location of a present tense accomplishment sentence will have to correspond exactly to the location of the event being described. Hence such sentences have the sense of narrating something in the vicinity just as it happens (*He shoots the ball!*)

Cooper goes on to discuss how locations other than the one where a sentence is actually uttered may become honorary utterance locations. This happens, for example, in the historical present or when someone narrates events they see on TV (following Ejerhed). Cooper seems correct in his claim that the variety of ways in which this occurs should not be a topic for formal semantic analysis; rather it seems to be understandable only in pragmatic or more general discourse analytic terms.

Past Tense. Every account of the past tense except those of Dowty and Parsons accommodates in some way the notion that past tense sentences are more definite than the usual tense logic operators. Even Dowty and Parsons, while claiming to treat the more fundamental use of the past tense, acknowledge the strength of the arguments that the past can refer to a definite time. Both cite Partee's example:

When uttered, for instance, half way down the turnpike such a sentence [as *I didn't turn off the stove*] clearly does not mean that there exists some time in the past at which I did not turn off the stove or that there exists no time in the past at which I turned off the stove.

There are, however, some sentences in which the past does seem completely indefinite. We can say, for example, *Columbus discovered America* or *Oswald killed Kennedy* without implying or presupposing anything about the date those events occurred beyond the fact that it was in the past. It

would be desirable to have an account of the past that could accommodate both the definite and indefinite examples. One solution, as discussed in Section 3.2, is that we interpret the past as a quantifier over a set of possible reference times.⁹ *I left the oven on* is true now only if the oven was left on at one of the past times I might be referring to. The context serves to limit the set of possible reference times. In the absence of contextual clues to the contrary the set comprises *all* the past times and the past is completely indefinite. In any case, the suggestion that the context determines a set of possible reference times seems more realistic than the suggestion that it determines a unique such time.

There is still something a little suspicious, however, about the notion that context determines a reference interval or a range of reference times for past tense sentences to refer to. One would normally take the ‘context of utterance’ to include information like the time and place the utterance is produced, the identity of the speaker and the audience, and perhaps certain other facts that the speaker and the audience have become aware of before the time of the utterance. But in this case it is clear that *Baltimore won the Pennant* and *Columbus discovered America* uttered in *identical* contexts would have *different* reference times.

A way out of the dilemma might be to allow the sentence itself to help identify the relevant components of a rich utterance context. Klein [1994] emphasizes the connection between the topic or background part of a sentence and its reference time (for him *topic time*). A full explanation of the mechanism will require taking into account the presupposition-focus structure of a sentence—that is, what new information is being communicated by the sentence. For example, when a teacher tells her class *Columbus discovered America*, the sentence would most naturally be pronounced with focal intonation on *Columbus*:

(18) COLUMBUS discovered America.

(19) ??Columbus discovered AMERICA.

??Columbus DISCOVERED America.

⁹The proposal is made in these terms in Kuhn [1979]. In Bennett–Partee the idea is rather that the reference time is an interval over whose subintervals the past tense quantifies. Thus the main difference between these accounts has to do with whether the reference time (or range of reference times) can be discontinuous. One argument for allowing it to be is the apparent reference to such times in sentences like *John came on a Saturday*. Another such argument might be based on the contention of Kuhn [1979] that the possible reference times are merely the times that happen to be maximally *salient* for speaker and audience. Vlach [1980] goes Partee–Bennett one further by allowing the past to indicate what obtains *in*, *at*, or *for* the reference interval.

The teacher is presupposing that someone discovered America, and communicating the fact that the discovery was made by Columbus. Similarly, when the teacher says *Bobby discovered the solution to problem number seven*, teacher and students probably know that Bobby was trying to solve problem number seven. The new information is that he succeeded. In those cases it is plausible to suppose that possible reference times would be the times at which the sentence's presupposition is true—the time of America's discovery and the times after which Bobby was believed to have started working on the problem. (As support for the latter claim consider the following scenario: Teacher assigns the problems at the beginning of class period. At the end she announces *Bobby discovered the solution to problem seven*. Susy objects *No he didn't. He had already done it at home.*)

A variety of theories have been proposed in recent years to explain how the intonational and structural properties of a sentence serve to help identify the presuppositions and 'new information' in a sentence.¹⁰ We will not go into the details of these here, but in general we can view a declarative sentence as having two functions. First, it identifies the relevant part of our mutual knowledge. Second, it supplies a new piece of information to be added to that part. It is the first function that helps delimit possible reference times. Previous discourse and non-linguistic information, of course, also play a role. When I say *Baltimore won the Pennant* it matters whether we have just been talking about the highlights of 1963 or silently watching this week's Monday Night Baseball.

Frequency. Bäuerle and von Stechow point out that interpreting the past tense as a quantifier ranging over possible reference times (or over parts of the reference time) makes it difficult to explain the semantics of frequency adverbs. Consider, for example, the sentence *Angelika sneezed exactly three times*, uttered with reference to the interval from two o'clock to three o'clock yesterday morning. We might take the sentence to mean that there are exactly three intervals between two and three with reference to which *Angelika sneezed* is true. But if *Angelika sneezed* means that she sneezed at least once within the time interval referred to, then whenever there is one such interval there will be an infinite number of them. So *Angelika sneezed exactly three times* could never be true. Alternatively we might take the sentence to mean that there was at least one time interval within which Angelika sneezed-three-times. But the intervals when Angelika sneezed three times will contain subintervals in which she sneezed twice. So in this case *Angelika sneezed exactly three times* would imply *Angelika sneezed exactly twice*.

This problem leads Bäuerle and von Stechow to insist that the past tense itself indicates simply that the eventuality described occupies that part of

¹⁰On the theory of focus, see for example Jackendoff, Rooth [1985; 1992], and Cresswell and von Stechow. On the nature of presupposition and factivity more generally, Levinson provides a good overview.

the reference time that lies in the past. On this interpretation, it does make sense to say that *Angelika sneezed three times* means that there were three times with reference to which *Angelika sneezed* is true. Tichý, using a different framework, arrives at a similar analysis. Unfortunately, this position also has the consequence that the simple sentence *Angelika sneezed*, taken literally, would mean that Angelika's sneeze lasted for the full hour between two and three. Bäuerle–von Stechow and Tichý both suggest that past tense sentences without explicit frequency operators often contain an *implicit* 'at least once' adverb. In a full treatment the conditions under which the past gets the added implicit adverb would have to be spelled out, so it is not clear how much we gain by this move. The alternative would seem to be to insist that the 'at least once' qualification is a normal part of the meaning of the tense which is dropped in the presence of frequency adverbs. This seems little better.

Vlach handles the frequency problem by allowing sentences to be true either 'in' or 'at' a time interval. *Angelika sneezed exactly three times* is true *at* the reference interval if it contains exactly three subintervals *at* which Angelika sneezes. On the other hand *Angelika sneezed* would normally be taken to assert that Angelika sneezed *in* the reference interval, i.e., that there is at least one time in the interval at which she sneezed. Again, a complete treatment would seem to require a way of deciding, for a given context and a given sentence, whether the sentence should be evaluated in or at the reference time.

We might argue that *all* the readings allowed by Vlach (or Bäuerle–von Stechow) are always present, but that language users tend to ignore the implausible ones—like those that talk about sneezes lasting two hours. But the idea that ordinary past tense sentences are riddled with ambiguities is not appealing.

The DRT analysis, on which frequency adverbs are examples of adverbs of quantification, can provide a somewhat more attractive version of the Bäuerle–von Stechow analysis. According to this view, *three times* binds the free time (or eventuality) variable present in the translation, as *always* did in (6)–(7) above. The situation is more straightforward when an additional temporal expression is present:

(20) On Tuesday, the bell rang three times.

(21) **three-times**_{*t*}(**past**(*t*) & **Tuesday**(*t*))(**rang**(the-bell,*t*)).

Here *Tuesday* helps to identify the set of times *three-times* quantifies over. **Tuesday**(*t*) indicates that *t* is a subinterval of Tuesday. A representation of this kind would indicate that there were three assignments of times during Tuesday to *t* at which the bell rang, where we say that the bell rang at *t* iff *t* is precisely the full interval of bell-ringing. The issue is more difficult when there is no restrictive argument for the adverb, as with *Angelika sneezed*

three times. One possibility is that it ranges over all past times. More likely, context would again provide a set of reference times to quantify over. In still other cases, as argued by Klein [1994], it ranges over times which are identified by the ‘background’ or presuppositions of the sentence. Thus, *Columbus sailed to AMERICA four times* means that, of the times when Columbus sailed somewhere, four were ones at which he sailed to America.

In terms of a DRT analysis, when there is no adverbial, as with *Angelika sneezed*, the temporal variable would be bound by whatever default process normally takes care of free variables (‘existential closure’ or another, as discussed above). This parallels the suggestion in terms of Bäuerle–von Stechow’s analysis, that ‘at least once’ is a component of meaning which is ‘dropped’ in the presence of an overt adverbial. Thus, in the DRT account there wouldn’t need to be a special stipulation for this.

There is still a problem with adverbials of duration, such as in *On Tuesday, the bell rang for five minutes*. This should be true, according to the above, if for some subinterval t of Tuesday, t is precisely the full time of the bell’s ringing and t lasts five minutes. Whether the sentence would be true if the bell in fact rang for ten minutes depends on whether *for five minutes* means ‘for at least five’ or ‘for exactly five’. If the former, the sentence would be true but inappropriate (in most circumstances), since it would generate an implicature that the bell didn’t ring for more than five minutes. If the latter, it would be false. It seems better to treat the example via implicature, since it is not as bad as *The bell rang for exactly five minutes* in the same situation, and the implication seems defeasible (*The bell rang for five minutes, if not more.*)

Future Tense. The architects of fragments of English with tense seem to have comparatively little to say about the future. Vlach omits it from his very comprehensive fragment, suggesting he may share Jespersen’s view that the future is not a genuine tense. Otherwise the consensus seems to be that the future is a kind of mirror image of the past with the exception, noted by Bennett and Partee, that the times to which the future can refer include the present. (Compare *He will now begin to eat* with *He now began to eat.*)

There appears to be some disagreement over whether the future is definite or indefinite. Tichý adopts the position that it is ambiguous between the two readings. This claim is difficult to evaluate. The sentence *Baltimore will win* can indicate either that Baltimore will win next week or that Baltimore will win eventually. But this difference can be attributed to a difference in the set of possible reference times as easily as to an ambiguity in the word *will*. It is of course preferable on methodological grounds to adopt a uniform treatment if possible.

3.4.2 Aspect

The Progressive. Those who wrote about the truth conditions of English tenses in the 1960's assumed that sentences were to be evaluated at instants of time. Dana Scott suggested (and Mongague [1970] seconds) a treatment of the present progressive according to which *Mary is swimming in the Channel* is true at an instant t if *Mary swims in the Channel* is true at every instant in an open interval that includes t . This account has the unfortunate consequence of making the present progressive form of a sentence imply its (indefinite) past. For a large class of sentences this consequence is desirable. If John is swimming in the Channel he did, at some very recent time, swim in the Channel. On the other hand there are many sentences for which this property does not hold. *John is drawing a circle* does not imply that John drew a circle. *Mary is climbing the Zugspitze* does not imply that Mary climbed the Zugspitze.

In Bennett–Partee, Vlach [1980] and Kuhn [1979] this difficulty avoided by allowing some present tense sentences to be evaluated at extended intervals of time as well as instants. *John is drawing a circle* means that the present instant is in the interior of an interval at which *John draws a circle* is true. The present instant can clearly be in such an interval even though *John drew a circle* is false at that instant. Sentences like *John swims in the Channel*, on the other hand, are said to have what Bennett and Partee label the *subinterval* property: their truth at an interval entails their truth at all subintervals of that interval. This stipulation guarantees that *Mary is swimming in the Channel* does imply *Mary swam in the Channel*.

Instantaneous events and gappy processes. Objections have been made to the Bennett–Partee analysis having to do with its application to two special classes of sentences. The first class comprises sentences that cannot plausibly be said to be true at extended intervals, but that do have progressive forms. Vlach, following Gilbert Ryle, calls these achievement sentences. We will follow Gabbay–Moravcsik and Bach in calling them instantaneous event sentences. They include *Baltimore wins*, *Columbus reaches North America*, *Columbus leaves Portugal* and *Mary starts to sweat*. It seems clear that instantaneous event sentences fail all the tests for statives. But if they are really true only instantaneously then the interval analysis would predict that they would never form true progressives.

The second class contains just the sentences whose present progressive implies their indefinite past. These are the *process* sentences. The Bennett–Partee analysis (and its modalized variation discussed below) have the consequence that process sentences can't have 'gappy' progressives. If *I sat in the front row of the Jupiter theater* was true at the interval from two o'clock to four o'clock last Saturday afternoon, then *I was sitting in the front row of the Jupiter theater* was true at all instants between those times including, perhaps, some instants at which I was really buying popcorn. This accord-

ing to Vlach, Bennett, and Gabbay–Moravcsik, is a conclusion that must be avoided.¹¹

Vlach's solution to the problems of instantaneous events and gappy processes is to give up the idea that a uniform treatment of the progressive is possible. For every non-stative sentence A , according to Vlach, we understand a notion Vlach calls the *process of* A or, simply $\text{proc}(A)$. The present progressive form of A simply says that our world is now in the state of $\text{proc}(A)$'s going on.

The nature of $\text{proc}(A)$, however, depends on the kind of sentence A is. If A is a process sentence then $\text{proc}(A)$ is 'the process that goes on when A is true.' For the other non-stative sentences, $\text{proc}(A)$ is a process that 'leads to' the truth of A , i.e., a process whose 'continuation... would eventually cause A to become true.' In fact, Vlach argues, to really make this idea precise we must divide the non-process, non-stative sentences into at least four subclasses.

The first subclass contains what we might (following Bach) call extended event sentences. Paradigm examples are *John builds a house* and *Mary swims across the Channel*. If an extended event sequence is true at an interval I then $\text{proc}(A)$ starts at the beginning of I and ends at the end of I . For the second subclass (*John realizes his mistake*, *Mary hits on an idea*) proc is not defined at all. For the third class (*Mary finishes building the house*, *Columbus reaches North America*) the progressive indicates that the corresponding process is in its final stages. For the fourth class (*Max dies*, *The plane takes off*) proc must give a process that culminates in a certain state.

Vlach's account is intended only as a rough sketch. As Vlach himself acknowledges, there remain questions of clarification concerning the boundaries of the classes of sentences and the formulation of the truth conditions. Furthermore, Vlach's account introduces a new theoretical term. If the account is to be really enlightening we would like to be sure that we have an understanding of proc that is independent of, but consistent with, the truth conditions of the progressive. Even if all the questions of clarification were resolved, Vlach's theory might not be regarded as particularly attractive because it abandons the idea of a uniform account of the progressive. Not even the sources of irregularity are regular. The peculiarity of the truth conditions for the progressive form of a sentence A are explained sometimes by the peculiarity of A 's truth conditions, sometimes by the way proc operates on A and sometimes by what the progressive says about $\text{proc}(A)$. In

¹¹This argument is not completely decisive. It would seem quite natural to tell a friend one meets at the popcorn counter *I am sitting in the front row*. On the other hand, if one is prepared to accept *I am not sitting in the front row* at popcorn buying time, then perhaps one should be prepared to accept *I sat in the front row before I bought the popcorn and again after*. This would suggest the process went on *twice* during the long interval rather than at one time with a gap.

this sense, Vlach's account is *pessimistic*. Other attempts have been made to give a more uniform account of the progressive. These *optimistic* theories may be divided into two groups depending on whether they propose that the progressive has a modal semantics.

Non-Modal Accounts. The analysis of Bennett-Partee discussed above was the first optimistic account presented developed in the formal semantic tradition. Since that time, two other influential non-modal proposals have been put forth. One is by Michael Bennett [1981] and one by Terence Parsons [1985; 1990]. The accounts of Vlach, Bennett and Parsons (and presumably anyone else) must distinguish between statives and non-statives because of the differences in their ability to form progressives. Non-statives must be further divided between processes and events if the inference from present progressive to past is to be selectively blocked. But in the treatments of Bennett and Parsons, as opposed to that of Vlach, all the differences among these three kinds of sentences are reflected in the untensed sentences themselves. Tenses and aspects apply uniformly.

Bennett's proposal is extremely simple.¹² The truth conditions for the present perfect form of *A* (and presumably all the other forms not involving progressives) require that *A* be true at a *closed* interval with the appropriate location. The truth conditions for the progressive of *A* require that *A* be true in an open interval with the appropriate location. Untensed process sentences have two special properties. First, if a process sentence is true at an interval, it is true at all closed subintervals of that interval. Second, if a process sentence is true at every instant in an interval (open or closed) then it is true at that interval. Neither of these conditions need hold for event sentences. Thus, if *John is building a house* is true, there must be an open interval at which *John builds a house* is true. But if there is no *closed* interval of that kind, then *John has built a house* will be false. On the other hand, *Susan is swimming* does imply *Susan has (at some time) swum* because the existence of an open interval at which *Susan swims* is true guarantees the existence of the appropriate closed intervals.

If this proposal has the merit of simplicity, it has the drawback of seeming very *ad hoc*—‘a logician's trick’ as Bennett puts it. Bennett's explanatory remarks are helpful. Events have a beginning and an end. They therefore occupy closed intervals. Processes, on the other hand, need not. But a process is composed, at least in part, of a sequence of parts. If Willy walks then there are many subintervals such that the eventualities described by *Willy walks* are also going on at these intervals. Events, however, need not be decomposable in this way.

The account offered by Parsons turns out to be similar to Bennett's. Parson's exposition seems more natural, however, because the metaphysical underpinnings discussed above are exposed. Parsons starts with the

¹²Bennett attributes the idea behind his proposal to Glen Helman.

assumption that there are three kinds of eventualities: *states*, *processes*, and *events*. Eventualities usually have *agents* and sometimes *objects*. An agent may or may not be *in* a state at a time. Processes may or may not be *going on* at a time. Events may or may not be *in development* at a time. In general, if e is an eventuality, we say that e *holds* at time t if the agent of e is in e at t or e is in development or going on at t . In addition, events can have the property of *culminating* at a time. The set of times at which an event holds is assumed to be an open interval and the time, if any, at which it culminates is assumed to be the least upper bound of the times at which it holds.

The structure of language mirrors this metaphysical picture. There are three kinds of untensed sentences: stative, process sentences and event sentences. Tensed sentences describe properties of eventualities. Stative and process sentences say that an eventuality *holds* at a time. Event sentences say that an eventuality *culminates* at a time. So, for example, *John sleeps* can be represented as (22) and *Jill bought a cat* as (23):

$$(22) \quad \exists e \exists t [\text{pres}(t) \wedge \text{sleeping}(e) \wedge \text{holds}(e, t) \wedge \text{agent}(e, \text{john})]$$

$$(23) \quad \exists e \exists t \exists x [\text{past}(t) \wedge \text{buying}(e) \wedge \text{culm}(e, t) \wedge \text{agent}(e, \text{jill}) \wedge \text{cat}(x) \wedge \text{obj}(e, x)].$$

The treatment of progressives is remarkably simple. Putting a sentence into the progressive has no effect whatsoever, other than changing the sentence from a non-stative into a stative. This means that, for process sentences, the present and progressive are equivalent. *John swims* is true if and only if *John is swimming* is true. Similarly, *John swam* is true if and only if *John was swimming* is true. For event sentences, the change in classification does affect truth conditions. *John swam across the Channel* is true if the event described *culminated* at some past time. *John was swimming across the Channel*, on the other hand, is true if the state of John's swimming across the Channel *held* at a past time. But this happens if and only if the event described by *John swims across Channel* was *in development* at that time. So it can happen that *John was swimming across the Channel* is true even though John never got to the other side.

Landman [1992] points out a significant problem for Parsons' theory. Because it is a purely extensional approach, it predicts that *John was building a house* is true if and only if there is a house x and a past event e such that e is an event of John building x and e holds. This seems acceptable. But Landman brings up examples like *God was creating a unicorn (when he changed his mind)*. This should be true iff there is a unicorn x and a past event e such that e is an event of God creating x and e holds. But it may be that the process of creating a unicorn involves some mental planning or magic words but doesn't cause anything to appear until the last moment,

when all of a sudden there is a fully formed unicorn. Thus no unicorn need ever exist for the sentence to be true. Landman's problem arises because of Parsons' assumption that eventualities are described primarily by the verb alone, as a swimming, drawing, etc., and by thematic relations connecting them to individuals, as **agent**(*e*, **jill**) or **obj**(*e*, *x*). There is no provision for more complex descriptions denoting a property like 'house-building'. The question is how intrinsic this feature is to Parsons' analysis of tense and aspect. One could adjust his semantics of verbs to make them multi-place intensional relations, so that *John builds a house* could be analyzed as:

$$(24) \quad \exists e \exists t [\text{pres}(t) \wedge \text{building}(e, \text{john}, \text{a house}) \wedge \text{culm}(e, t)].$$

But then we must worry about how the truth conditions of **building**(*e*, **john**, **a house**) are determined on a compositional basis and how one knows what it is for an eventuality of this type to hold or culminate. However, while the challenge is real, it is not completely clear that it is impossible to avoid Landman's conclusion that the progressive cannot be treated in extensional terms.

It seems likely that, with the proper understanding of theoretical terms, Parsons, Vlach, and Bennett could be seen as saying very similar things about the progressive. Parsons' exposition seems simpler than Vlach's, however, and more natural than Bennett's. These advantages may have been won partly by reversing the usual order of analysis from ordinary to progressive forms. Vlach's account proceeds from *A* to *proc*(*A*) to the state of *proc*(*A*)'s holding. In Bennett's, the truth conditions for the progressive of *A* are explained in terms of those for *A*. If one compares the corresponding progressive and non-progressive forms on Parson's account, however, one sees that in the progressive of an event sentence, something is *subtracted* from the corresponding non-progressive form. The relations between the progressive and non-progressive forms seem better accommodated by viewing events as processes plus culminations rather than by viewing processes as eventualities 'leading to' events.

On the other hand the economy of Parsons' account is achieved partly by ignoring some of the problems that exercise Vlach. The complexity of Vlach's theory increases considerably in the face of examples like *Max is dying*. To accommodate this kind of case Parsons has two options. He can say that they are ordinary event sentences that are in development for a time and then culminate, or he can say that they belong to a new category—achievement—of sentences that culminate but never hold. The first alternative doesn't take account of the fact that such eventualities can occur at an instant (compare *Max was dying and then died at 5:01* with *Jane was swimming across the Channel and then swam across the Channel at 5:01*). The second requires us to say that the progressive of these sentences, if it can be formed at all, involves a 'change in meaning' (cf. Parsons [1990,

p. 24, 36]). But the progressive *can* be formed and spelling out the details of the meaning changes involved will certainly spoil some of Parsons' elegance.

Unfinished progressives and Modal Accounts. According to the Bennett–Partee account of progressives, *John was building a house* does not imply that John built a house. It does, however, imply that John will eventually have built a house. Yet it seems perfectly reasonable to say:

(25) John was building a house when he died.

One attempt to modify the account to handle this difficulty is given by Dowty [1979]. Dowty's proposal is that we make the progressive a *modal* notion.¹³ The progressive form of a sentence *A* is true at a time *t* in world *w* just in case *A* is true at an interval containing *t* in all worlds *w'* such that *w'* and *w* are exactly alike up to *t* and the course of events after *t* develops in the way most compatible with past events. The *w'*-worlds mentioned are referred to as 'inertia worlds'. (25) means that *John builds a house* is eventually true in all the worlds that are inertia worlds relative to ours at the interval just before John's death.

If an account like this is to be useful, of course, we must have some understanding of the notion of inertia world independent of its role in making progressive sentences true. The idea of a development maximally compatible with past events may not be adequate here. John's death and consequent inability to finish his house may have been natural, even inevitable, at the time he was building it. In Kuhn [1979] the suggestion is that it is the expectations of the language users that are at issue. But this seems equally suspect. It is quite possible that because of a bad calculation, we all mistakenly expect a falling meteor to reach earth. We would not want to say in this case that the meteor *is* falling to earth.

Landman attempts to identify in more precise terms the alternate possible worlds which must be considered in a modal semantics of the progressive. We may label his the *counterfactual analysis*, since it attempts to formalize the following intuition: Suppose we are in a situation in which John fell off the roof and died, and so didn't complete the house, though he would have finished it if he hadn't died. Then (25) is true *because* he would have finished if he hadn't died. Working this idea out requires a bit more complexity, however. Suppose not only that John fell off the roof and died, but also that if he hadn't fallen, he would have gotten ill and not finished the house anyway. The sentence is still true, however, and this is because he would have finished the house if he hadn't fallen and died and hadn't gotten ill. We can imagine still more convoluted scenarios, where other dangers lurk for John. In the end, Landman proposes that (25) is true iff *John builds a*

¹³Dowty attributes this idea to David Lewis.

house would be true if nothing were to interrupt some activity that John was engaged in.

Landman formalizes his theory in terms of the notion of the *continuation branch* of an event e in a world w . He assumes an ontology wherein events have *stages* (cf. Carlson [1977]); the notion of ‘stage of an eventuality’ is not defined in a completely clear way. Within a single world, all of the temporally limited subeventualities of e are stages of e . An eventuality e' may also be a stage of an eventuality e in another world. It seems that this can occur when e' is duplicated in the world of e by an eventuality which is a stage of e . The continuation branch of e in w , $C(e, w)$, is a set of event-world pairs; $C(e, w)$ contains all of the pairs $\langle a, w \rangle$ where a is a stage of e in w . If e is a stage of a larger event in some other possible world, we say that it *stops* in w (otherwise it simply ends in w). If e stops in w at time t , the continuation branch moves to the world w_1 most similar to w in which e does not stop at t . Suppose that e_1 is the event in w_1 of which e is a stage; then all pairs $\langle a, w_1 \rangle$, where a is a stage of e_1 in w_1 , are also in $C(e, w)$. If e_1 stops in w_1 , the continuation branch moves to the world most similar to w_1 in which e_1 does not stop, etc. Eventually, the continuation branch may contain a pair $\langle e_n, w_n \rangle$ where a house gets built in e_n . Then the continuation branch ends. We may consider the continuation branch to be the maximal extension of e . *John was building a house* is true in w iff there is some event in w whose continuation branch contains an event of John building a house.

Landman brings up one significant problem for his theory. Suppose Mary picks up her sword and begins to attack the whole Roman army. She kills a few soldiers and then is cut down. Consider (26):

(26) Mary was wiping out the Roman army.

According to the semantics described above, (26) ought to be true. Whichever soldier actually killed Mary might not have, and so the continuation branch should move to a world in which he didn’t. There some soldier kills Mary but might not have, so . . . Through a series of counterfactual shifts, the continuation branch of Mary’s attack will eventually reach a world in which she wipes out the whole army. Landman assumes that (26) ought not be true in the situation envisioned. The problem, he suggests, is that the worlds in which Mary kills a large proportion of the Roman army, while possible, are outlandishly unreasonable. He therefore declares that only ‘reasonable worlds’ may enter the continuation branch.

Landman’s analysis of the progressive is the most empirically successful optimistic theory. Its major weaknesses are its reliance on two undefined terms: *stage* and *reasonable*. The former takes part in the definition of when an event stops, and so moves the continuation branch to another world. How do we know with (as) the event John was engaged in didn’t end when he

died? Lots of eventualities did end there; we wouldn't want to have *John was living to be 65* to be true simply because he would probably have lived that long if it weren't for the accident. We know that the construction event didn't end because we know it was supposed to be a house-building. Thus, Landman's theory requires a primitive understanding of when an event is complete, ending in a given world, and when it is not complete and so may continue on in another world. In this way, it seems to recast in an intensional theory Parsons' distinction between holding and culminating. The need for a primitive concept of reasonableness of worlds is perhaps less troubling, since it could perhaps be assimilated to possible worlds analyses of epistemic modality; still, it must count as a theoretical liability.

Finally, we note that Landman's theory gives the progressive a kind of interpretation quite different from any other modal or temporal operator. In particular, since it is nothing like the semantics of the perfect, the other aspect we will consider, one wonders why the two should be considered members of a common category. (The same might be said for Dowty's theory, though his at least resembles the semantics for modalities.)

The Perfect. Nearly every contemporary writer has abandoned Montague's position that the present perfect is a completely indefinite past. The current view (e.g. [McCoard, 1978; Richards, 1982; Mittwoch, 1988]) seems to be that the time to which it refers (or the range of times to which it might refer) must be an *Extended Now*, an interval of time that begins in the past and includes the moment of utterance. The event described must fall somewhere within this interval. This is plausible. When we say *Pete has bought a pair of shoes* we normally do not mean just that a purchase was made at some time in the past. Rather we understand that the purchase was made recently. The view also is strongly supported by the observation that the present perfect can always take temporal modifiers that pick out intervals overlapping the present and never take those that pick out intervals entirely preceding the present: *Mary has bought a dress since Saturday*, but not **Mary has bought a dress last week*. These facts can be explained if the adverbials are constrained to have scope over the perfect, so that they would have to describe an extended now.

There is debate, however, about whether the extended now theory should incorporate two or even three readings for the perfect. The uncontroversial analysis, that suggested above, locates an event somewhere within the extended now. This has been called the *existential use*. Others have argued that there is a separate *universal* or *continuative use*. Consider the following, based on some examples of Mittwoch:

- (27) Sam has lived in Boston since 1980.

This sentence is compatible with Sam's still living in Boston, or with his having come, stayed for a while, and then left. Both situations are com-

patible with the following analysis: the extended now begins in 1980, and somewhere within this interval Sam lives in Boston. However, supporters of the universal use (e.g. [McCawley, 1971; Mittwoch, 1988; Michaelis, 1994]) argue that there is a separate reading which requires that Sam's residence in Boston continue at the speech time: (27) is true iff Sam lives in Boston throughout the whole extended now which begins in 1980.

Michaelis argues that the perfect has a third reading, the *resultative use*. A resultative present perfect implies that there is a currently existing result state of the event alluded to in the sentence. For example, *John has eaten poison* could be used to explain the fact that John is sick. Others (McCawley [1971], Klein [1994]) argue that such cases should be considered examples of the existential use, with the feeling that the result is especially important being a pragmatic effect. At the least one may doubt analyses in terms of result state on the grounds that precisely which result is to be focused on is never adequately defined. Any event will bring about some new state, if only the state of the event having occurred, and most will bring about many. So it is not clear how this use would differ in its truth conditions from the existential one.

Stump argues against the Extended Now theory on the basis of the occurrence of perfects in nonfinite contexts like the following (his Chapter IV, (11); cf. McCoard, Klein, Richards who note similar data):

- (28) Having been on the train yesterday, John knows exactly why it derailed.

Stump provides an analysis of the perfect which simply requires that no part of the event described be located after the evaluation time. In a present perfect sentence, this means that the event can be past or present, but not future. Stump then explains the ungrammaticality of **Mary has bought a dress last week* in pragmatic terms. This sentence, according to Stump, is truth conditionally equivalent to *Mary bought a dress last week*. Since the latter is simpler and less marked in linguistic terms, the use of the perfect should implicate that the simple past is inappropriate. But since the two are synonymous, it cannot be inappropriate. Therefore, the present perfect with a definite past adverbial has an implicature which can never be true. This is why it cannot be used (cf. Klein [1992] for a similar explanation).

Klein [1992; 1994] develops a somewhat different analysis of perfect aspect from those based on interval semantics. He concentrates on the relevance of the aspectual classification of sentences for understanding different 'uses' of the perfect. He distinguishes *0-state*, *1-state*, and *2-state* clauses: A 0-state clause describes an unchanging state of affairs (*The Nile is in Africa*); a 1-state sentence describes a state which obtains at some interval while not obtaining at adjoining intervals (*Peter was asleep*); and a 2-state clause denotes a change from one lexically determined state to another (*John opened*

the window). Here, the first state (the window's being closed) is called the *source state*, and the second (the window's being open) the *target state*. He calls the maximal intervals which precede and follow the interval at which a state holds its *pretime* and *posttime* respectively.

Given this framework, Klein claims that all uses of the perfect can be analyzed as the reference time falling into the posttime of the most salient situation described by the clause. Since the states described by 0-state sentences have no posttime, the perfect is impossible (**The Nile has been in Africa*). With 1-state sentences, the reference time will simply follow the state in question, so that *Peter has been asleep* will simply indicate that Peter has at some point slept ('experiential perfect'). With 2-state sentences, Klein stipulates that the salient state is the source state, so that *John has opened the window* literally only indicates that the reference time (which in this case corresponds to the utterance time) follows a state of the window being closed which itself precedes a state of the window being open. It may happen that the reference time falls into the target state, in which case the window must still be open ('perfect of result'); alternatively, the reference time may follow the target state as well—i.e. it may be a time after which the window has closed again—giving rise to another kind of experiential perfect.

One type of case which is difficult for Klein is what he describes as the 'perfect of persistent situation', as in *We've lived here for ten years*. This is the type of sentence which motivated the universal/continuative semantics within the Extended Now theory. In Klein's terms, here it seems that the reference time, the present, falls into the state described by a 1-state sentence, and not its posttime. Klein's solution is to suggest that the sentence describes a state which is a substate of the whole living-here state, one which comprises just the first ten years of our residency, a 'living-here-for-ten-years' state. The example indicates that we are in the posttime of this state, a fact which does not rule out that we're now into our eleventh year of living here. On the other hand, such an explanation does not seem applicable to other examples, such as *We've lived here since 1966*.

Existence presuppositions. Jespersen's observation that the present perfect seems to presuppose the present existence of the subject in cases where the past tense does not has been repeated and 'explained' many times. We are now faced with the embarrassment of a puzzle with too many solutions. The contemporary discussion begins with Chomsky, who argues that *Princeton has been visited by Einstein* is all right, but *Einstein has visited Princeton* is odd. James McCawley points out that the alleged oddity of the latter sentence actually depends on context and intonation. Where the existence presupposition does occur, McCawley attributes it to the fact that the present perfect is generally used when the present moment is included in an interval during which events of the kind being described *can* be true.

Thus, *Have you seen the Monet exhibition?* is inappropriate if the addressee is known to be unable to see it. (*Did you* is appropriate in this case.) *Frege has contributed a lot to my thinking* is appropriate to use even though Frege is dead because Frege *can* now contribute to my thinking. *My mother has changed my diapers many times* is appropriate for a talking two year old, but not for a normal thirty year old. *Einstein has visited Princeton* is odd because Einsteinian visits are no longer possible. *Princeton has been visited by Einstein* is acceptable because Princeton's being visited *is* still possible.

In Kuhn [1983] it is suggested that the explanation may be partly syntactic. Existence presuppositions can be canceled when a term occurs in the scope of certain operators. Thus *Santa is fat* presupposes that Santa exists, but *According to Virginia, Santa is fat* does not. There are good reasons to believe that past and future apply to sentences, whereas perfect applies only to intransitive verb phrases. But in that case it is natural that presuppositions concerning the subject that do hold in present perfect sentences fail in past and future sentences.

Guenthner requires that at least one of the objects referred to in a present perfect sentence (viz., the topic of the sentence) must exist at utterance time. Often, of course, the subject will be the topic.

The explanation given by Tichý is that, in the absence of an explicit indication of reference time, a present perfect generally refers to the lifetime of its subject. If this does not include the present, then the perfect is inappropriate.

Overall, the question of whether these explanations are compatible, and whether they are equally explanatory, remains open.

3.4.3 Tense in Subordinate Clauses

The focus in all of the preceding discussion has been on occurrences of tense in simple sentences. A variety of complexities arise when one tries to accommodate tense in subordinate clauses. Of particular concern is the phenomenon known as *Sequence of Tense*. Consider the following:

(29) John believed that Mary left.

(30) John believed that Mary was pregnant.

Example (29) says that at some past time t John had a belief that at some time $t' < t$, Mary left. This reading is easily accounted for by a classic Priorean analysis: the time of evaluation is shifted into the past by the first tense operator, and then shifted further back by the second. (30), which differs from (29) in having a stative subordinate clause, has a similar reading, but has another as well, the so-called 'simultaneous reading', on which the time of Mary's alleged pregnancy overlaps with the time of John's belief. It would seem that the tense on *was* is not semantically active. A

traditional way of looking at things is to think of the tense form of *was* as triggered by the past tense of *believed* by a morphosyntactic sequence of tense (SOT) rule. Following Ogihara [1989; 1995], we could formalize this idea by saying that a past tense in a subordinate clause governed by another past tense verb is deleted prior to the sentence's being interpreted. For semantic purposes, (30) would then be *John believed that Mary be (tenseless) pregnant*. Not every language has the SOT rule. In Japanese, for example, the simultaneous reading of (30) would be expressed with present tense in the subordinate clause.

The SOT theory does not explain why simultaneous readings are possible with some clauses and not with others. The key distinction seems to be between states and non-states. One would hope to be able to relate the existence of simultaneous readings to the other characteristic properties of statives discussed in Section 3.4.1 above.

Sentences like (31) pose special problems. One might expect for it to be equivalent to either (30), on the simultaneous reading, or (32).

(31) John believed that Mary is pregnant.

(32) John believed that Mary would now be pregnant.

A simultaneous interpretation would be predicted by a Priorian account, while synonymy with (32) would be expected by a theory which said that present tense means 'at the speech time'. However, as pointed out by Enç [1987], (31) has a different, problematical interpretation; it seemingly requires that the time of Mary's alleged pregnancy extend from the belief time up until the speech time. She labels this the *Double Access Reading* (DAR). Recent theories of SOT, in particular those of Ogihara [1989; 1995] and Abusch [1991; 1995], have been especially concerned with getting a correct account of such 'present under past' sentences.

Enç's analysis of tense in intensional contexts begins with the proposal that tense is a referential expression. She suggests that the simultaneous interpretation of (30) should be obtained through a 'binding' relationship between the two tenses, indicated by coindexing as in (33). The connection is similar to that holding with nominal anaphora, as in (34).

(33) John PAST₁ believed that Mary PAST₁ was pregnant.

(34) John₁ thinks that he₁ is smart.

This point of view lets Enç say that both tense morphemes have a usual interpretation. Her mechanisms entail that all members of a sequence of coindexed tense morphemes denote the same time, and that each establishes the same temporal relationship as the highest ('first') occurrence. Ogihara elucidates the intended interpretation of structures like (33) by translating them into Intensional Logic.

$$(35) \quad t_1 < s^* \& \text{believe}'(t_1, j, ^\wedge [t_1 < s^* \wedge \text{be-pregnant}(t_1, m)]).$$

Here s^* denotes the speech time. If the two tenses were not coindexed, as in (36), the second would introduce $t_2 < t_1$ to the translation:

$$(36) \quad \text{John PAST}_1 \text{ believed that Mary PAST}_2 \text{ was pregnant.}$$

$$(37) \quad t_1 < s^* \& \text{believe}'(t_1, j, ^\wedge [t_2 < t_1 \wedge \text{be-pregnant}(t_2, m)]).$$

This represents the non-simultaneous ('shifted') reading.

Accounting for the DAR is more complex. Enç proposes that there need to be two ways that temporal expressions may be linked. Expressions receive pairs of indices, so that with a configuration $A_{\langle i, j \rangle} \dots B_{\langle k, l \rangle}$, if $i = k$, then A and B refer to the same time, while if $j = l$, then the time if B is included in that of A . The complement clause *that Mary is pregnant* is then interpreted outside the scope of the past tense. The present tense is linked to the speech time. As usual, however, the two tenses may be coindexed, but only via their second indices. This gives us something like (38).

$$(38) \quad \exists x(x = [\text{Mary PRES}_{\langle 0, 1 \rangle} \text{ be pregnant}] \text{ John PAST}_{\langle 2, 1 \rangle} \text{ believes } x).$$

This representation says that Mary is pregnant at the speech time and that the time of John's belief is a subinterval of Mary's pregnancy. Thus it encodes the DAR.

The mechanisms involved in deriving and interpreting (38) are quite complicated. In addition, examples discussed by Abusch [1988], Baker [1989] and Ogihara [1995] pose a serious difficulty for Enç's view.

$$(39) \quad \text{John decided a week ago that in ten days at breakfast he would say to his mother that they were having their last meal together.}$$

Here, on the natural interpretation of the sentence, the past tense of *were* does not denote a time which is past with respect to either the speech time or any other time mentioned in the sentence. Thus it seems that the tense component of this expression cannot be semantically active.

As mentioned above, Ogihara proposes that a past tense in the right relation with another past tense may be deleted from a sentence prior to semantic interpretation. (Abusch has a more complex view involving feature passing, but it gets similar effects.) This would transform (39) into (40).

$$(40) \quad \text{John PAST decided a week ago that in ten days at breakfast he } \emptyset \text{ will say to his mother that they } \emptyset \text{ be having their last meal together.}$$

Notice that we have two deleted tenses (marked ‘ $\emptyset$ ’) here. *Would* has become tenseless *woll*, a future operator evaluated with respect to the time of the deciding. Then breakfast time ten days after the decision serves as the time of evaluation for *he say to his mother that they be having their last meal together*. Since there are no temporal operators in this constituent, the time of the saying and that of the last meal are simultaneous.

The double access sentence (31) is more difficult story. Both Ogihara and Abusch propose that the DAR is actually a case of *de re* interpretation, similar to the famous Ortcutt examples of Quine [1956]. Consider example (31), repeated here:

(31) John believed that Mary is pregnant.

Suppose John has glimpsed Mary two months ago, noticing that she is quite large. At that time he thought ‘Mary is pregnant’. Now you and I are considering why Mary is so large, and I report John’s opinion to you with (31). The sentence could be paraphrased by *John believed of the state of Mary’s being large that it is a state of her being pregnant*. (Abusch would frame this analysis in terms of a *de re* belief about an interval, rather than a state, but the difference between these two formulations appears slight.) Both Ogihara and Abusch give their account in terms of the analysis of *de re* belief put forward by Lewis [1979] and extended by Cresswell and von Stechow [1982]. These amount to saying that (31) is true iff the following conditions are met: (i) John stands in a suitable *acquaintance relation* *R* to a state of Mary’s (such as her being large), in this case the relation of having glimpsed it on a certain occasion, and (ii) in all of John’s belief-worlds, the state to which he stands in relation *R* is a state of Mary being pregnant.

A *de re* analysis of present under past sentences may hope to give an account of the DAR. Suppose we have an analysis of tense whereby the present tense in (31) entails that the state in question holds at the speech time. Add to this the fact that the acquaintance relation, that John had glimpsed this state at the time he formed his belief, entails that the state existed already at that time. Together these two points require that the state stretch from the time of John’s belief up until the speech time. This is the DAR.

The preceding account relies on the acquaintance relation to entail that the state have existed already at the past time. The idea that it would do so is natural in light of Lewis’ suggestion that the relation must be a causal one: in this case that John’s belief has been caused, directly or indirectly, by the state. However, as Abusch [1995] points out, there is a problem with this assumption: it sometimes seems possible to have a future-oriented acquaintance relation. Consider Abusch’s example (41) (originally due to Andrea Bonomi).

- (41) Leo will go to Rome on the day of Lea's dissertation. Lia believes that she will go to Rome with him then.

Here, according to Abusch, we seem to have a *de re* attitude by Lia towards the future day of Lea's dissertation. Since the acquaintance relation cannot be counted on to require in (31) that the time of Mary's being large overlaps the time when John formed his belief, both Abusch and Ogihara have had to introduce extra stipulations to serve this end. But at this point the explanatory force of appealing to a *de re* attitude is less clear.

There are further reasons to doubt the *de re* account, at least in the form presented. Suppose that we're wondering whether the explanation for Mary's appearance is that she's pregnant. John has not seen Mary at all, but some months ago her mother told John that she is, he believed her, and he reported on this belief to me. It seems that I could say (31) as evidence that Mary is indeed pregnant. In such a case it seems that the sentence is about the state *we're* concerned with, not one which provided John's evidence.

3.4.4 *Tense and discourse*

One of the major contributions of DRT to the study of tense is its focus on 'discourse' as the unit of analysis rather than the sentence. Sentential analyses treat reference times as either completely indeterminate or given by context. In fact the 'context' that determines the time a sentence refers to may just be the sentences that were uttered previously. Theorists working within DRT have sought to provide a detailed understanding of how the reference time of a sentence may depend on the tenses of the sentence and its predecessors.

As mentioned above, DRS's will include events, states, and times as objects in the universe of discourse and will specify relations of precedence and overlap among them. Precisely which relations hold depends on the nature of the eventualities being described. The key distinction here is between 'atelic' eventualities (which include both states and processes) and 'telic' ones. Various similar algorithms for constructing DRS's are given by Kamp, Kamp and Rohrer, Hinrichs, and Partee, among others. Let us consider the following pair of examples:

- (42) Mary was eating a sandwich. Pedro entered the kitchen.

- (43) Pedro went into the hall. He took off his coat.

In (42), the first sentence describes an atelic eventuality, a process, whereas the second describes a telic event. The process is naturally taken to temporally contain the event. In contrast, in (43) both sentences describe telic events, and the resulting discourse indicates that the two happened in sequence.

A DRS construction procedure for these two could work as follows: With both the context provides an initial past reference time r_0 . Whenever a past tense sentence is uttered, it is taken to temporally coincide with the past reference time. A telic sentence introduces a new reference time that follows the one used by the sentence, while an atelic one leaves the reference time unchanged. So, in (42), the same reference time is used for both sentences, implying temporal overlap, while in (43) each sentence has its own reference time, with that for the second sentence following that for the first.

Dowty [1986a] presents a serious critique of the DRT analysis of these phenomena. He points out that whether a sentence describes a telic or atelic eventuality is determined by compositional semantics, and cannot be read off of the surface form in any direct way. He illustrates with the pair (44)–(45).

(44) John walked. (activity)

(45) John walked to the station. (accomplishment)

Other pairs are even more syntactically similar (*John baked a cake* vs. *John baked cakes*.) This consideration is problematical for DRT because that theory takes the unit of interpretation to be the entire DRS. A complete DRS cannot be constructed until individual sentences are interpreted, since it must be determined whether sentences describe telic or atelic eventualities before relations of precedence and overlap are specified. But the sentences cannot be interpreted until the DRS is complete.

Dowty proposes that the temporal sequencing facts studied by DRT can be accommodated more adequately within interval semantics augmented by healthy amounts of Gricean implicature and common-sense reasoning. First of all, individual sentences are compositionally interpreted within a Montague Grammar-type framework. Dowty [1979] has shown how differences among states, processes, and telic events can be defined in terms of their temporal properties within interval semantics. (For example, as mentioned above, A is a stative sentence iff, if A is true at interval I , then A is true at all moments within I .) The temporal relations among sentence are specified by a single, homogeneous principle, the *Temporal Discourse Interpretation Principle* (TDIP), which states:

- (46) **TDIP** Given a sequence of sentences $S_1, S_2, \dots, S_n$ to be interpreted as a narrative discourse, the reference time of each sentence S_i (for i such that $1 < i \leq n$) is interpreted to be:
- (a) a time consistent with the definite time adverbials in S_i , if there are any;
 - (b) otherwise, a time which immediately follows the reference time of the previous sentence S_{i-1} .

Part (b) is the novel part of this proposal. It gives the same results as DRT in all-telic discourses like (43), but seems to run into trouble with atelic sentences like the one in (42). Dowty proposes that (42) really does describe a sequence of a process or state of Mary eating a sandwich followed by an event of Pedro entering the kitchen; this is the literal contribution of the example (Nerbonne [1986] makes a similar proposal.) However, common sense reasoning allows one to realize that a process of eating a sandwich generally takes some time, and so the time at which Mary was actually eating a sandwich might have started some time before the reference time and might continue for some time afterwards. Thus (42) is perfectly consistent with Mary continuing to eat the sandwich while Pedro entered the kitchen. In fact, Dowty would suggest, in normal situations this is just what someone hearing (42) would be likely to conclude.

Dowty's analysis has an advantage in being able to explain examples of inceptive readings of atelic sentences like *John went over the day's perplexing events once more in his mind. Suddenly, he was fast asleep. Suddenly* tells us that the state of being asleep is new. World knowledge tells us that he could not have gone over the days events in his mind if he were asleep. Thus the state must begin after the event of going over the perplexing events in his mind. DRT would have a more difficult time with this example; it would have to propose that *be asleep* is ambiguous between an atelic (state) reading and a telic (achievement) reading, or that the word *suddenly* cancels the usual rule for atelics.

As Dowty then goes on to discuss, there are a great many examples of discourses in which the temporal relations among sentences do not follow the neat pattern described by the DRT algorithms and the TDIP. Consider:

- (47) Mary did the dishes carefully. She filled the sink with hot water. She added a half cup of soap. Then she gently dipped each glass into the sudsy liquid.

Here all of the sentences after the first one describe events which comprise the dish-washing. To explain such examples, an adherent of DRT must propose additional DRS construction procedures. Furthermore, there exists the problem of knowing which procedures to apply; one would need rules to determine which construction procedures apply before the sentences within the discourse are interpreted, and it is not clear whether such rules can be formulated in a way that doesn't require prior interpretation of the sentences involved. Dowty's interval semantics framework, on the other hand, would say that the relations among the sentences here are determined pragmatically, overriding the TDIP. The weakness of this approach is its reliance on an undeveloped pragmatic theory.

4 TENSE LOGICS FOR NATURAL LANGUAGE

4.1 *Motivations*

General surveys of tense logic are contained elsewhere in this Handbook (Burgess, Finger, Gabbay and Reynolds, and Thomason, all in this Volume). In this section we consider relations between tense logic and tense and aspect in natural language. Work on tense logic, even among authors concerned with linguistic matters, has been motivated by a variety of considerations that have not always been clearly delineated. Initially, tense logic seems to have been conceived as a generalization of classical logic that could better represent logical forms of arguments and sentences in which tense plays an important semantic role. To treat such items within classical logic requires extensive ‘paraphrase’. Consider the following example from Quine [1982]:

- (48) George *V* married Queen Mary, Queen Mary is a widow, therefore George *V* married a widow.

An attempt to represent this directly in classical predicate logic might yield

$$(48a) \quad Mgm, Wm \models \exists x(Mgx \wedge Wx),$$

which fallaciously represents it as valid. When appropriately paraphrased, however, the argument becomes something like:

- (49) Some time before the present is a time when George *V* married Queen Mary, Queen Mary is a widow at the present time, therefore some time before the present is a time at which George *V* married a widow,

which, in classical logic, is represented by the nonvalid:

$$(49a) \quad \exists t(Tt \wedge Btn \wedge Mgmt), Wmn \models \exists t(Tt \wedge Btn \wedge \exists x(Wxn \wedge Mgmt)).$$

If we want a logic that can easily be *applied* to ordinary discourse, however, such extensive and unsystematized paraphrase may be unsatisfying. Arthur Prior formulated several logical systems in which arguments like (48) could be represented more directly and, in a series of papers and books in the fifties and sixties, championed, chronicled and contributed to their development. (See especially [Prior, 1957; Prior, 1967] and [Prior, 1968].) A sentence like *Queen Mary is a widow* is not to be represented by a formula that explicitly displays the name of a particular time and that is interpreted simply as *true* or *false*. Instead it is represented as *Wm*, just as in (48), where such formulas are now understood to be true or false only relative to a time. Past

and future sentences are represented with the help of *tense logical operators* like those mentioned in previous sections. In particular, most of Prior's systems contained the past and future operators with truth conditions:

$$(50) \quad t \models \mathcal{P}A \text{ if and only if } \exists s(s < t \ \& \ s \models A)$$

$$t \models \mathcal{F}A \text{ if and only if } \exists s(t < s \ \& \ s \models A)$$

(where $t \models A$ means A is true at time t and $s < t$ means time s is before time t). This allows (48) to be represented:

$$(48b) \quad \mathcal{P}Mgm, Wm \models \mathcal{P}\exists x(Mgx \wedge Wx).$$

Quine himself thought that a logic to help prevent us misrepresenting (48) as (48a) would be 'needlessly elaborate'. 'We do better,' he says, 'to make do with a simpler logical machine, and then, when we want to apply it, to paraphrase our sentences to fit it.' In this instance, Quine's attitude seems too rigid. The advantages of the simpler machine must be balanced against a more complicated paraphrase and representation. While (49a) may represent the form of (49), it does not seem to represent the form of (48) as well as (48b) does. But if our motivation for constructing new tense logics is to still better represent the logical forms of arguments and sentences of natural language, we should be mindful of Quine's worries about their being needlessly elaborate. We would not expect a logical representation to capture all the nuances of a particular tense construction in a particular language. We would expect a certain economy in logical vocabulary and rules of inference.

Motivations for many new systems of tense logic may be seen as more semantical than logical. A semantics should determine, for any declarative sentence S , context C , and possible world w , whether the thought expressed when S is uttered in C is true of w . As noted in previous sections, the truth conditions associated with Prior's $\mathcal{P}$ and $\mathcal{F}$ do not correspond very closely to those of English tenses. New systems of tense logic attempt to forge a closer correspondence. This might be done with the view that the tense logic would become a convenient *intermediary* between sentences of natural language and their truth conditions. That role was played by tensed intensional logic in Montague's semantics. An algorithm translates English sentences into formulas of that system and an inductive definition specifies truth conditions for the formulas. As noted above, Montague's appropriation of the Priorean connectives into his intensional logic make for a crude treatment of tense, but refined systems might serve better. Specifications of truth conditions for the tensed intensional logic (and, more blatantly for the refined tense logics), often seem to use a first order theory of temporal precedence (or containment, overlap, etc.) as yet another intermediary. (Consider clauses (50) above, for example.) One may wonder, then, whether it wouldn't be

better to skip the first intermediary and translate English sentences directly into such a first order theory. Certainly the most perspicuous way to give the meaning of a particular English sentence is often to ‘translate’ it by a formula in the language of the first order theory of temporal precedence, and this consideration may play a role in some of the complaints against tense logics found, for example, in [van Benthem, 1977] and [Massey, 1969]. Presumably, however, a *general* translation procedure could be simplified by taking an appropriate tense logic as the target language.

There is also another way to understand the attempt to forge a closer correspondence between tense logical connectives and the tense constructions of natural language. We may view tense logics as ‘toy’ languages, which, by isolating and idealizing certain features of natural language, help us to understand them. On this view, the tense logician builds models or simulations of features of language, rather than parts of linguistic theories. This view is plausible for, say Kamp’s logic for ‘now’ and Galton’s logic of aspect (see below), but it is difficult to maintain for more elaborate tense logics containing many operators to which no natural language expressions correspond.

Systems of tense logic are sometimes defended against classical first order alternatives on the grounds that they don’t commit language users to an ontology of temporal moments, since they don’t explicitly quantify over times. This defense seems misguided on several counts. First, English speakers do seem to believe in such an ontology of moments, as can be seen from their use of locutions like ‘at three o’clock sharp’. Second, it’s not clear what kind of ‘commitment’ is entailed by the observation that the language one uses quantifies over objects of a certain kind. Quine’s famous dictum, ‘to be is to be the value of a bound variable,’ was not intended to express the view that we are committed to what we quantify over in ordinary language, but rather that we are committed to what our best scientific theories quantify over, when these are cast in first order logic. There may be some weaker sense in which, by speaking English, we may be committing ourselves to the existence of entities like chances, sakes, average men and arbitrary numbers, even though we may not believe in these objects in any ultimate metaphysical sense. Perhaps we should say that *the language* is committed to such objects. (See Bach [1981].) But surely the proper test for this notion is simply whether the best interpretation of our language requires these objects: ‘to be is to be an element of a model.’ And, whether we employ tense logics or first order theories, our best models do contain (point-like and/or extended) times. Finally, even if one were sympathetic to the idea that the weaker notion of commitment was revealed by the range of first order quantifiers, there is reason to be suspicious of claims that a logic that properly models any substantial set of the temporal features of English would have fewer ontological commitments than a first order theory of temporal precedence. For, as Cresswell has argued in detail [1990; 1996],

the languages of such logics turn out to be equivalent in expressive power to the language of the first order theories. One might reasonably suppose in this case that the ontological commitments of the modal language should be determined by the range of the quantifiers of its first order equivalent. As discussed in Section 3.1, then, the proper defense of tense logic's replacement of quantifiers by operators is linguistic rather than metaphysical.

4.2 Interval based logics

One of the most salient differences between the traditional tense logical systems and natural language is that all the formulas of the former are evaluated at instants of time, whereas at least some of the sentences of the latter seem to describe what happens at extended temporal periods. We are accustomed to thinking of such periods as comprising continuous stretches of instants, but it has been suggested, at least since Russell, that extended periods are the real objects of experience, and instants are abstractions from them. Various recipes for constructing instants from periods are contained in Russell [1914], van Benthem [1991], Thomason [1984; 1989] and Burgess [1984]. Temporal relations among intervals are more diverse than those among instants, and it is not clear which of these relations should be taken as primitive for an interval based tense logic. Figure 4.2 shows 13 possible relations that an interval A can bear to the fixed interval B .

We can think of $<$ and $>$ as precedence and succession, $\ll$ and $\gg$ as immediate precedence and succession and $\subset$, $\supset$, and $\circ$ as inclusion, containment and overlap. The subscripts l and r are for 'left' and 'right'. Under reasonable understandings of these notions and reasonable assumptions about the structure of time, these can all be defined in elementary logic from precedence and inclusion. For example, $A \ll B$ can be defined by $A < B \wedge \neg \exists x(A < x \wedge x < B)$, and $A \circ_l B$ by $\exists x(x \subset A \wedge x < B) \wedge (\exists x)(x \subset A \wedge x \subset B) \wedge \exists x(x \subset A \wedge x > B)$. It does not follow, however, that a tense operator based on any of these relations can be defined from operators based on $<$ and $\subset$. Just as instant based tense logics include both $\mathcal{P}$ and $\mathcal{F}$ despite the fact that $>$ is elementarily definable from $<$, we may wish to include operators based on a variety of the relations above in an interval based tense logic. For each of the relations R listed in the above chart, let $[R]$ and $\langle R \rangle$ be the box and diamond operators defined with R as the accessibility relation. (We are presupposing some acquaintance with the Kripke semantics for modal logics here. See Bull and Segerberg in this *Handbook* for background.) Then $\langle < \rangle$ and $\langle > \rangle$ are interval analogs of Prior's $\mathcal{P}$ and $\mathcal{F}$, and $\langle \subset \rangle$ is a connective that Dana Scott suggested as a rough analog of the progressive. Halpern and Shoham [1986] (and Shoham [1988]) point out that if we take the three converse pairs $[\ll]$ and $[\gg]$, $[\subset_l]$ and $[\supset_l]$ and $[\subset_r]$ and $[\supset_r]$ as primitive we can give simple definitions of the connectives associated with the remaining relations:

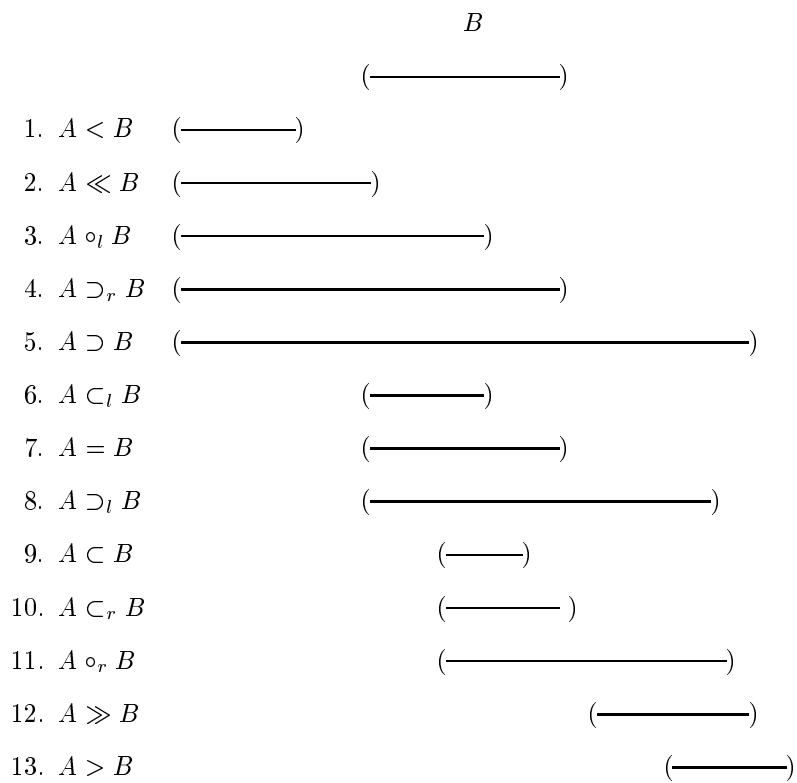


Figure 2.

$$[<] = [\ll][\ll] \quad [>] = [\gg][\gg]$$

$$[\supset] = [\supset_l][\supset_r] \quad [\subset] = [\subset_l][\subset_r]$$

$$[\circ_l] = [\supset_l][\subset_r] \quad [\circ_r] = [\supset_r][\subset_l]$$

If it is assumed that intervals always contain durationless atoms, i.e., subintervals s such that $\neg\exists t(t \subset s)$, then Venema shows that we can do better. For then $[\supset_l] \perp$ and $[\supset_r] \perp$ will be true only at atoms, and there are formulas $[l]A = (A \wedge [\supset_l] \perp) \vee \langle \supset_l \rangle (A \wedge [\supset_l] \perp)$ and $[r]A = (A \wedge [\supset_r] \perp) \vee \langle \supset_r \rangle (A \wedge [\supset_r] \perp)$ saying that A is true at the left and right ‘endpoints’ of an interval. $[\ll]$ and $[\gg]$ can now be defined by $[r][\subset_l]$ and $[l][\subset_r]$. (The assumption that there are durationless ‘intervals’ undercuts the idea that instants are mere abstractions, but it seems appropriate for linguistic applications of tense logic, since language users do, at some level, presume the existence of both intervals and instants.)

Call the tense-logical language with operators $[\subset_l], [\supset_l], [\subset_r]$ and $[\supset_r]$, HSV in honor of its inventors. Since HSV can so easily express all the relations on the table above, one might expect it to be sufficient to express any temporal relations that common constructions in natural language do. As Venema shows, however, there are limitations to its expressive power. Consider the binary connective $\wedge^*$ such that $(s, t) \models (A \wedge^* B)$ iff, for some $r, s < r < t, (s, r) \models A$ and $(r, t) \models B$. Lloyd Humberstone argues that $\wedge^*$ is the tense logical connective that properly expresses temporal conjunction, i.e., *and* in the sense of *and next*. But no formula in HSV can express $\wedge^*$. Further, as Venema shows, there is a sense in which this expressive poverty is unavoidable in interval logics. Call a model $M = (I, \subset_l, \subset_r, \supset_l, \supset_r, V)$ for HSV ‘instant generated’ if there is some nonempty set T ordered by $<$ such that I is the set of all $(x, y) \in (T \times T)$ for which $x \leq y$, and $\subset_l, \subset_r, \supset_l$ and $\supset_r$ are the appropriate relations on I . (For example $(r, s) \supset_l (u, v)$ iff $u = r$ and $s > v$.) Instant generated HSV-models, then, are models in which formulas are evaluated at pairs of indices, i.e., they are two-dimensional models. The truth conditions for the connectives determine a translation that maps formulas of HSV to ‘equivalent’ formulas in predicate logic with free variables r and s . Similar translations could be obtained for any language in which the truth conditions of the connectives can be expressed in elementary logic. Venema shows, however, that for no finite set of connectives will this translation include in its range every formula with variable r and s . This result holds even when the equivalent formulas are required to agree only on models for which the instants form a dense linear order. This contrasts with a fundamental result in instant-based tense logics, that for dense linear orders, the two connectives ‘since’ and ‘until’ are sufficient to express everything that can be said in elementary logic with one free variable. (See Burgess [2001]).

Several authors have suggested that in tense logics appropriate for natural language there should be constraints on the set of intervals at which a formula can be true. The set $\|A\|_M$ of indices at which formula A is true in model M is often called the truth set of A . Humberstone requires that valuations be restricted so that truth sets of sentence letters be closed under containment. That ‘downward closure’ property seems natural for stative sentences (see Section 3.4.2). The truth of *The cat is on the mat* at the interval from two to two to two thirty apparently entails its truth at the interval from two ten to two twenty. But downward closure is not preserved under ordinary logical negation. If *The cat is on the mat* is true at $(2:00, 2:30)$ and all its subintervals, but not at $(1:30, 3:00)$ then $\neg(\textit{the cat is on the mat})$ is true at $(1:30, 3:00)$ but not all of its subintervals. Humberstone suggests a stronger form of negation, which we might call $[\neg]$. $[\neg]A$ is true at interval i if A is false at all subintervals of i . Such a negation may occur in one reading of *The cat isn’t on the mat*. It can also be used to express a more purely tense logical connective: $[\supset]$ can be defined as $[\neg][\neg]$. We obtain a reasonable tense logic by adding the standard past and future connectives $\langle \rangle$ and $\langle \rangle$.

Statives also seem to obey an upward closure constraint. If A is true in each of some sequence of adjoining or overlapping intervals, it is also true in the ‘sum’ of those intervals. Peter Röper observes that, in the presence of downward closure, upwards closure is equivalent to the condition that A is true in i if it is true ‘almost everywhere’ in i , i.e., if every subinterval of i contains a subinterval at which A is true. (See Burgess [1982a] for an interesting list of other equivalents of this and related notions.) Following Röper, we may call a truth set homogeneous if it satisfies both upwards and downwards closure. Humberstone’s strong negation preserves homogeneity, but the tense connectives $\langle \rangle$ and $\langle \rangle$ do not. For suppose the temporal intervals are the open intervals of some densely ordered set of instants, and A is true only at (s, t) and its subintervals. Then the truth set of A is homogeneous. But every proper subinterval of (s, t) verifies $\langle \rangle A$, and so every subinterval of (s, t) contains a subinterval that verifies the formula, whereas (s, t) itself does not verify the formula, and so the truth set of $\langle \rangle A$ is not homogeneous. To ensure that homogeneity is preserved, Röper replaces the standard truth conditions for the future operator by a condition stating that $\langle \rangle A$ is true at i if every subinterval of i contains a subinterval i' such that A is true at some $w > i'$. The past operator is similarly altered. This ensures that all formulas have homogeneous truth sets and the resulting system admits a simple axiomatization. One may wonder whether the future and past tenses of statives really are themselves statives in natural language, and thus whether homogeneity really ought to be preserved. But if one is thoroughgoing (as Humberstone and Röper seem to be, but Venema does not), about the attitude that (extended) intervals are the genuine temporal objects, then it does seem reasonable to suppose that for stative A , $A \rightarrow$

$\langle \rangle A$ and $A \rightarrow \langle \rangle A$ are logical truths. If the cat is on the mat, then, if one looks sufficiently close to the present, it will be on the mat, and if one looks sufficiently close in the other direction, it was on the mat. For otherwise we would have to believe that the present was the instant at which it came or left. Indeed, the ‘present implies past’ property was cited by Aristotle (in *Metaphysics* IX) as a distinguishing feature of ‘energaie,’ a category that surely includes the statives. The formulas $A \rightarrow \langle \rangle A$ and $A \rightarrow \langle \rangle A$ are not theorems of HSV or standard tense logics unless $<$ is reflexive, but they are theorems of Röper’s homogeneous interval tense logic.

4.3 ‘Now’, ‘then’, and keeping track of times

Another way in which natural language differs from Priorean tense logics is its facility in conveying that the eventualities described in various scattered clauses of a sentence obtain simultaneously. Consider first an example in which exterior and interior clauses describe what obtains at the moment of utterance.

(51) This is 1996 and one day everyone now alive will be dead.

If we represent this as $P \wedge \forall x(Lx \rightarrow \mathcal{F}Dx)$, we fail to imply that those alive today will all be dead at a common future moment. If we pull the future operator outside the quantifier, we get $P \wedge \mathcal{F}\forall x(Lx \rightarrow Dx)$, which wrongly implies that there will be a time when live people are (simultaneously) dead. A solution (following Kamp [1971] and Prior [1968]) is to evaluate formulas at pairs of times, the first of which ‘keeps track’ of the moment of utterance and the second of which is used to evaluate expressions inside tense operators. $(s, t) \models A$ can be understood as asserting that A is true at t when part of an expression uttered at s . The truth conditions for the Priorean operators use the second coordinate: $(s, t) \models \mathcal{P}A$ iff $\exists t' < t(s, t') \models A$ and $(s, t) \models \mathcal{F}A$ iff $\exists t' > t(s, t') \models A$. A new connective $\mathcal{N}$ corresponding to the adverb *now* is added satisfying $(s, t) \models \mathcal{N}A$ iff $(s, s) \models A$. Validity in a model is to be understood as truth whenever uttered, i.e., $M \models A$ iff for every time t in M , $(t, t) \models A$. On this understanding $A \leftrightarrow \mathcal{N}A$ is valid, so it may appear that $\mathcal{N}$ is vacuous. Its effect becomes apparent when it appears within the scope of the other tense operators. $\mathcal{P}(A \leftrightarrow \mathcal{N}A)$, for example, is false when A assumes a truth value at utterance time that differs from the value it had until then. This condition can still be expressed without the new connective by $(A \wedge \neg \mathcal{P}A) \vee (\neg A \wedge \neg \mathcal{P}\neg A)$, and in general, as Kamp shows, $\mathcal{N}$ is eliminable in propositional Priorean tense logics. If the underlying language has quantifiers, however, $\mathcal{N}$ does increase its expressive power. For example, the troublesome example above can be represented as

(51a) $P \wedge \mathcal{F}\forall x(\mathcal{N}Lx \rightarrow Dx)$.

The new connective can be used to ensure that embedded clauses get evaluated after the utterance moment as well as simultaneously with it. Consider Kamp's

- (52) A child was born who will be king.

To represent this as $\mathcal{P}(A \wedge \mathcal{F}B)$ would imply only that the child is king after its birth. To capture the sense of the English *will*, that the child is king after the utterance moment, we need $\mathcal{P}(A \wedge \mathcal{N}\mathcal{F}B)$.

Vlach [1973] shows that in a somewhat more general setting $\mathcal{N}$ can be used to cause evaluation of embedded clauses at still other times. Take the sentence *It is three o'clock and soon Jones will cite all those who are now speeding*, which has a structure like (51), and put it into the past:

- (53) It was three o'clock and Jones would soon cite those who were then speeding.

We cannot represent this by simply applying a past operator to (51a) because the resulting formula would imply that Jones was going to ticket those who were speeding at the time of utterance. Vlach suggests we add an 'index' operator to the language with truth conditions very similar to $\mathcal{N}$'s

$$(s, t) \models \mathcal{I}A \text{ iff } (t, t) \models A.$$

If an $\mathcal{N}$ occurs within the scope of an $\mathcal{I}$ it can be read as *then*. This allows, for example, the sentence (44) to be represented as

$$\mathcal{P}\mathcal{I}(P \wedge \forall x(\mathcal{N}Sx \rightarrow Cx).$$

In general, if A contains no occurrence of $\mathcal{I}$, the utterance time is 'fixed' in the sense that the truth value of A at $\langle u, t \rangle$ depends on the truth values of its subformulas at pairs $\langle u, t' \rangle$. The occurrence of an $\mathcal{I}$ 'shifts' the utterance time so that evaluating A at $\langle u, t \rangle$ may require evaluating the subformulas that are within the scope of the $\mathcal{I}$ at pairs $\langle u', t' \rangle$ for u' different than u .

With Kamp's *now*, we can keep track of the utterance time and one other time. With Vlach's *then*, we still track two times, although neither need coincide with utterance. Several authors have suggested that a tense-logical system adequate to represent natural language must allow us to keep track of more than two times. The evidence is not entirely convincing, but it has motivated some interesting revisions in the Priorean framework. Gabbay [1974; 1976] points to examples like the following:

(54) John said he would come.

(55) Ann will go to a school her mother attended and it will become better than Harvard,

which, he maintains, have interpretations suggested by the formulas

(54a) $\exists t_1 < t_0 (\text{John says at } t_1 \text{ that } \exists t_2 (t_1 < t_2 < t_0 \wedge \text{John comes at } t_2))$

(55a) $\exists t_1 > t_0 \exists s (s \text{ is a school Ann goes to } s \text{ at } t_1 \wedge \exists t_2 < t_0 (\text{Ann's mother goes to } s \text{ at } t_2 \wedge \exists t_3 > t_1 (s \text{ is better than Harvard at } t_3)))$.

Saarinen's exhibits include

(56) Every man who ever supported the Vietnam War believes now that one day he will have to admit that he was an idiot then,

interpreted as

(56a) $\forall x (x \text{ is a man} \rightarrow \forall t_1 < t_0 (x \text{ supports the Vietnam War at } t_1) (x \text{ believes at } t_0 \text{ that } \exists t_2 > t_0 (x \text{ has to admit at } t_2 \text{ that } x \text{ is an idiot at } t_1)))$,

and (57) Joe said that a child had been born who would become ruler of the world,

which, Saarinen argues, has at least the two readings

(57a) $\exists t < t_0 (\text{Joe says at } t \text{ that } \exists s < t \exists x (\text{Child } x \wedge \text{Born } xs \wedge \exists u > s \text{ Ruler } xu)))$

(57b) $\exists t < t_0 (\text{Joe says at } t \text{ that } \exists s < t \exists x (\text{Child } x \wedge \text{Born } xs \wedge \exists u > t \text{ Ruler } xu)))$

according to whether the sentence reported is *A child was born who would become ruler*, or *A child was born who will become ruler*. (Note that the sequence of tense theories discussed in Section 3.4.3 above conflict with the readings proposed here for (54) and (57).)¹⁴

Cresswell [1990] points to examples of a more explicitly quantificational form:

¹⁴They hold that requirement in (54a) that t_2 precede t_0 is not part of the truth conditions for (54) (though it may be implicated). Similarly, they hold that (57a) is the sole reading of (57).

- (58) There will be times such that all persons now alive will be A_1 at the first or A_2 at the second or... A_n at the n th.
- (58a) $\exists t_1 \dots \exists t_n (t_0 < t_1 \wedge \dots \wedge t_0 < t_n \wedge \forall x (x \text{ is alive at } t_0 \rightarrow (x \text{ is } A_1 \text{ at } t_1 \vee \dots \vee x \text{ is } A_n \text{ at } t_n)))$.

Some of the troublesome examples could be expressed in a Priorean language. For example, for (55) we might propose:

- (55b) $\exists s (\text{SCHOOL}(s) \wedge \mathcal{P} \text{ATTEND}(\text{ann's mother}, s) \wedge \mathcal{F}(\text{ATTEND}(\text{ann}, s) \wedge \mathcal{F} \text{BETTER}(s, \text{harvard})))$

But as a toy version of (55) or the result of applying a uniform English-to-tense-logic translation procedure, this may seem implausible. It requires a reordering of the clauses in (55), which removes *that her mother attended* from inside the scope of the main tense operator. Other troublesome examples can be represented with the help of novel two-dimensional operators. For example, Gabbay suggests that the appropriate reading of (54) might be represented $\mathcal{P} \text{Johnsaythat} \mathcal{F}_2 A$, where $\langle u, t \rangle \models \mathcal{F}_2 A$ iff either $t < u$ and $\exists s (t < s < u \wedge \langle u, s \rangle \models A)$ or $u < t$ and $\exists s (u < s < t \wedge \langle u, s \rangle \models A)$. (A variety of other two dimensional tense operators are investigated in Åqvist and Guenther ([1977; 1978]). This approach, however, seems somewhat *ad hoc*. In the general case, Gabbay argues, “we must keep record of the entire sequence of points that figure in the evaluation of a formula] and not only that, but also keep track of the kind of operators used.”

We sketch below five more general solutions to the problem of tracking times. Each of these introduces an interesting formal system in which the times that appear at one stage in the evaluation of a formula can be remembered at later stages, but none of these seems to provide a fully accurate model of the time-tracking mechanisms of natural language.

4.3.1 Backwards-looking operators (Saarinen)

Add to the language of tense logic a special ‘operator functor’ $\mathbf{D}$. For any operator $\square$, $\mathbf{D}(\square)$ is a connective that ‘looks back’ to the time at which the preceding $\square$ was evaluated. For example, (47) can be represented

- (56b) $\forall x (x \text{ is a man} \rightarrow \neg \mathcal{P} \neg (x \text{ supported the Vietnam war} \rightarrow \mathbf{D}(\mathcal{P})(x \text{ believesthat } \mathcal{F}(x \text{ hastoadmitthat } \mathbf{D}(\mathbf{D}(\mathcal{P}))(x \text{ is an idiot}))))))$

if we have the appropriate **believesthat** and **hastoadmitthat** operators. Within a more standard language,

- (59) $A \wedge \mathcal{F}(B \wedge \mathcal{P}(C \wedge \mathcal{F}(\mathbf{D} \wedge \mathbf{D}(\mathcal{P})E) \wedge \mathbf{D}(\mathcal{F})F)$

is true at w iff $\exists x \exists y \exists z (w < x, y < x, y < z, w \models A, x \models B, y \models C, z \models D, x \models E \text{ and } y \models F)$. In this example $\mathbf{D}(\mathcal{P})$ and $\mathbf{D}(\mathcal{F})$ ‘look back’ to the times at which the preceding $\mathcal{P}$ and $\mathcal{F}$ were evaluated, namely, x and y . This condition can be expressed without the backwards operators by

$$(59a) \quad A \wedge \mathcal{F}(B \wedge E \wedge \mathcal{P}(C \wedge F \wedge \mathcal{F}D)),$$

but (as with (55b)) this requires a reordering of the clauses, and (as with (56b)) the reordering may be impossible in a richer formal language. It is a little hard to see how the semantics for $\mathbf{D}$ might be made precise in Tarski-style truth definition. Saarinen suggests a game-theoretic interpretation, in which each move is made with full knowledge of previous moves. Iterated $\mathbf{D}(\Box)$ ’s look back to more distant $\Box$ ’s so that, for example,

$$A \wedge \mathcal{P}(B \wedge \mathcal{F}(C \wedge \mathcal{F}(D \wedge \mathbf{D}(\mathcal{F})\mathbf{D}(\mathcal{F})E) \wedge \mathbf{D}(\mathcal{P})F))$$

is true at w iff $\exists x \exists y \exists z (x < w, x < y < z, w \models A, x \models B, y \models C, z \models D, x \models E \text{ and } w \models A)$. Logics based on this language would differ markedly from traditional ones. For example, if time is dense $\mathcal{F}A \rightarrow \mathcal{F}\mathcal{F}A$ is valid when A does not contain $\mathbf{D}$ ’s, but not when A is of the form $\mathbf{D}(\mathcal{F})B$.

4.3.2 Dating sentences (Blackburn [1992; 1994])

Add a special sort of sentence letters, each of which is true at exactly one moment of time. Blackburn thinks of these as naming instants and calls his systems ‘nominal tense logics,’ but they are more accurately viewed as ‘dating sentences’, asserting, for example *It is now three pm on July 1, 1995*. Tense logical systems in this language can be characterized by adding to the usual tense logical axioms the schema

$$n \wedge \mathcal{E}(n \wedge A) \rightarrow A$$

where n is a dating sentence and $\mathcal{E}$ is any string of $\mathcal{P}$ ’s and $\mathcal{F}$ ’s. In place of (59), we can now write:

$$(59b) \quad A \wedge \mathcal{F}(B \wedge \mathbf{i} \mathcal{P}(C \wedge \mathbf{j} \wedge \mathcal{F}D)) \wedge \mathcal{P}\mathcal{F}(\mathbf{i} \wedge E) \wedge \mathcal{P}\mathcal{F}(\mathbf{j} \wedge F).$$

Here $\mathbf{i}$ and $\mathbf{j}$ ‘date’ the relevant times at which B and C are true, so that the truth of $\mathbf{i} \wedge E$ and $\mathbf{j} \wedge F$ requires the truth of E and F at those same times.

4.3.3 Generalization of $\mathcal{N}$ - $\mathcal{I}$ (Vlach [1973, appendix])

To the language of Priorean tense logic, add connectives $\mathcal{N}_i$ and $\mathcal{I}_i$ for all non-negative integers i . Let formulas be evaluated at pairs (s, i) where $s = (s_0, s_1, \dots)$ is an infinite sequence of times and i is a non-negative integer, specifying the coordinate of s relevant to the evaluation. $\mathcal{N}_i A$ indicates that

A is to be evaluated at the time referred to when $\mathcal{I}_i$ was encountered. More precisely,

$$(s, i) \models \mathcal{P}A \text{ iff } \exists t < s_i((s_0, \dots, s_{i-1}, t, s_{i+1}, \dots), i) \models A$$

$$(s, i) \models \mathcal{F}A \text{ iff } \exists t < s_i((s_0, \dots, s_{i-1}, t, s_{i+1}, \dots), i) \models A$$

$$(s, i) \models \mathcal{I}_j A \text{ iff } ((s_0, \dots, s_{j-1}, s_i, s_{j+1}, \dots), i) \models A$$

$$(s, i) \models \mathcal{N}_j A \text{ iff } (s, j) \models A$$

The truth of sentence letters at (s, i) depend only on s_i and formulas are to be considered valid in a model if they are true at all pairs $((t, t, \dots), 0)$. In this language (59) can be expressed

$$(59c) \quad A \wedge \mathcal{F}\mathcal{I}_1(B \wedge \mathcal{P}\mathcal{I}_2(C \wedge \mathcal{F}(D \wedge \mathcal{N}_2 E \wedge \mathcal{N}_1 F))).$$

Here $\mathcal{I}_1$ and $\mathcal{I}_2$ ‘store’ in s_1 and s_2 the times at which B and C are evaluated and $\mathcal{N}_2$ and $\mathcal{N}_1$ shift the evaluation to s_2 and s_1 , causing F and E to be evaluated at times there stored.

4.3.4 The backspace operator (Vlach [1973, appendix])

Add to the language of Priorean tense logic a single unary connective **B**. Let formulas be evaluated at finite (nonempty) sequences of times according to the conditions:

$$(t_1, \dots, t_n) \models \mathcal{P}A \text{ iff } \exists t_{n+1} < t_n((t_1, \dots, t_{n+1}) \models A)$$

$$(t_1, \dots, t_n) \models \mathcal{F}A \text{ iff } \exists t_{n+1} > t_n((t_1, \dots, t_{n+1}) \models A)$$

$$(t_1, \dots, t_{n+1}) \models \mathbf{B}A \text{ iff } (t_1 \dots, t_n) \models A$$

$$(\text{and, if } n = 0, (t_1) \models \mathbf{B}A \text{ iff } (t_1) \models A)$$

The truth value of sentence letters depends only on the last time in the sequence, and formulas are considered valid in a model when they are true at all length-one sequences. (59) is now represented

$$(59d) \quad A \wedge \mathcal{F}(B \wedge \mathcal{P}(C \wedge \mathcal{F}(D \wedge \mathbf{B} E \wedge \mathbf{B} B F))).$$

The indices of evaluation here form a stack. In the course of evaluating a formula a new time is pushed onto the stack whenever a Priorean tense connective is encountered and it is popped off whenever a **B** is encountered. Thus, **B** is a ‘backspace’ operator, which causes its argument to be evaluated at the time that had been considered in the immediately preceding stage of evaluation. In terms of this metaphor, Kamp’s original ‘now’ connective

was, in contrast, a ‘return’ operator, causing its argument to be evaluated at the time that was given at the initial moment of evaluation.

4.3.5 Generalization of $\mathcal{N}\text{--}\mathcal{I}$ (Cresswell [1990])

Generalize the language of Vlach’s $\mathcal{N}\text{--}\mathcal{I}$ system just as in solution 3. Let formulas be evaluated at infinite sequences of times and let the truth definition contain the following clauses:

$$(s_0, s_1, s_2, \dots) \models \mathcal{P}A \text{ iff } \exists s < s_0 ((s, s_1, s_2, \dots) \models A)$$

$$(s_0, s_1, s_2, \dots) \models \mathcal{F}A \text{ iff } \exists s > s_0 ((s, s_1, s_2, \dots) \models A)$$

$$(s_0, s_1, \dots, s_i, \dots) \models \mathcal{I}_i A \text{ iff } (s_0, s_1, \dots, s_{i-1}, s_0, s_{i+1}, \dots) \models A$$

$$(s_0, s_1, \dots, s_i, \dots) \models \mathcal{N}_i A \text{ iff } (s_i, s_1, s_2, \dots) \models A$$

A formula is considered valid if it is true at all constant sequences $(s, s, \dots)$. Then we can express (59) above as:

$$(59e) \quad A \wedge \mathcal{FI}_1(B \wedge \mathcal{PI}_2(C \wedge \mathcal{F}(D \wedge \mathcal{N}_2 E \wedge \mathcal{N}_1 F))).$$

As in solution 3, $\mathcal{I}_1$ and $\mathcal{I}_2$ store in s_1 and s_2 the times at which B and C are evaluated. Subsequent occurrences of $\mathcal{N}_2$ and $\mathcal{N}_1$ restore those times to s_0 so that E and F can be evaluated—with respect to them.

Each of the systems described in 4.3.1–4.3.5 has a certain appeal, and we believe that none of them has been investigated as thoroughly as it deserves. We confine ourselves here to a few remarks about their expressive powers and their suitability to represent tense constructions of natural language. Of the five systems, only Cresswell’s $\mathcal{N}\text{--}\mathcal{I}$ generalization permits atomic formulas to depend on more than one time. This makes it possible, for example, to represent *Johnson ran faster than Lewis*, meaning that Johnson ran faster in the 1996 Olympics than Lewis did in the 1992 Olympics, by Rmn . We understand R to be a predicate (*runs faster than*) which, at every pair of times, is true or false of pairs of individuals. Since the issues involved in these representations are somewhat removed from the ones discussed here, and since the other systems could be generalized in this way if desired, this difference is not significant. If we stipulate that the truth value of a sentence letter at s in Cresswell’s system depends only on s_0 then, for each of the systems, there is a translation of formulas into the classical first order language with identity and a countable collection of temporally monadic predicates and a single temporally dyadic predicate $<$ (and, in the case of nominal tense logic, a countable collection of temporal constants). We say ‘temporally’ monadic and dyadic because, if the base language of these systems is the language of predicate logic, it will already

contain polyadic predicates that apply to tuples of *individuals*. The translation maps these to predicates with an additional temporal argument, and it maps tense formulas with free individual variables into classical formulas with those same free variables and additional free temporal variables. The sentential version of Cresswell's $\mathcal{N}$ - $\mathcal{I}$ provides an example. Associate with each sentence letter p a unary predicate letter p^τ and fix two (disjoint) sequences of variables $x_0, x_1, \dots$ and $y_0, y_1, \dots$. A translation τ from Cresswell-formulas into classical formulas is defined by the following clauses (where A^x/y is the result of replacing all free occurrences of y in A by x):

$$\text{i) } \tau p = p^\tau x_0$$

$$\text{ii) } \tau \mathcal{P}A = \exists y < x_0 (\tau A)^y /_{x_0}, \text{ where } y \text{ is the first } y_i \text{ that does not occur in } \tau A$$

$$\text{iii) } \tau \mathcal{F}A = \exists y > x_0 (\tau A)^y /_{x_0}, \text{ where } y \text{ is as above}$$

$$\text{iv) } \tau \mathcal{I}_j A = (\tau A)^{x^0} /_{x_j}$$

$$\text{v) } \tau \mathcal{N}_j A = (\tau A)^{x^j} /_{x_0}$$

To every model M for Cresswell's language there corresponds a classical model M' with the same domain which assigns to each predicate letter p^τ the set of times at which p is true in M . τA expresses A in the sense that $(s_0, s_1, \dots) \models_M A$ iff τA is true in M' under the assignment that assigns s_i to x_i for $i = 0, 1, \dots$. Viewing M and M' as the same model, we can say that a tense-logical formula expresses a classical one when the two formulas are true in the same models. (Of course in defining a tense-logical system, we may restrict the class of appropriate models. By 'true in the same models' we mean true in the same models appropriate for the tense logic.) A formula with one free variable in the first order language with unary predicates and $<$ might be called a 'classical tense'. From the translation above we may observe that every Cresswell formula in which each occurrence of a connective $\mathcal{N}_j$ lies within the scope of an occurrence of $\mathcal{I}_j$ expresses a classical tense. If every classical tense is expressible in tense-logical system, the system is said to be temporally complete.

An argument in Chapter IV of Cresswell establishes that, as long as $<$ is assumed to be connected (so that quantification over times can be expressed in the tense language), every classical tense without $<$ can be expressed in his generalization of the $\mathcal{N}$ - $\mathcal{I}$ language. It is not difficult to see that this holds as well for Vlach's generalization. For consider the following translation τ mapping Cresswell's system into Vlach's:

$\tau A = \mathcal{N}_0 A$ if A is a sentence letter,

$\tau \mathcal{P} A = \mathcal{P} \tau A$,

$\tau \mathcal{F} A = \mathcal{F} \tau A$,

$\tau \mathcal{I}_i A = \mathcal{I}_i \tau A$,

$\tau \mathcal{N}_i A = \mathcal{I}_x \mathcal{I}_{x+1} \dots \mathcal{I}_{2x} \mathcal{N}_i \mathcal{I}_{x-i} \mathcal{N}_{x-i} \tau A$ where x is the successor of the least integer greater than every subscript that occurs in $\mathcal{N}_i A$.

Then, using the subscripts C and V for Vlach's system and Cresswell's, $\mathbf{s} \models_C A$ iff $(\mathbf{s}, 0) \models_V \tau A$. So, if A is a classical tense without $<$, there is a formula A_C that expresses A in Cresswell's system, and τA_C will express A in Vlach's system.

The question of whether every classical tense is expressible is more difficult. As we saw with Kamp's $\mathcal{N}$, questions about expressive power are sensitive to the underlying language. $\mathcal{N}$ adds nothing to the expressive power of sentential tense logic, but it does add to the expressive power of predicate tense logic. The examples suggest that the same is true of the backwards-looking and backspace operators. A well known result of Kamp (see Burgess [2001]) states that, if time is like the reals, every tense can be expressed with the connectives U (*until*) and S (*since*) with truth conditions $U(A, B)$ iff $\exists t > t_0 (t \models A \wedge \forall s (t_0 < s < t \rightarrow s \models B))$ and $S(A, B)$ iff $\exists t < t_0 (t \models A \wedge \forall s (t < s < t_0 \rightarrow s \models B))$. By constructing a pair of models that can be discriminated by formulas with U and S but not by any Priorean formulas, one can show that Priorean tense logic is not temporally complete. A reduction of the sentential backwards-looking and backspace systems to the ordinary ones, therefore, would imply their temporal incompleteness. From the pairs of ordinary models that are indistinguishable by Priorean formulas, we can easily construct pairs that are indistinguishable in the language of Blackburn's dating sentences. (Pick corresponding times t and t' in the two models and require that every dating sentence be true exactly at t in the first model and exactly at t' in the second.) So that system also fails to be temporally complete.¹⁵

For a number of reasons, the suitability of a system of tense logic for natural language should not be identified with its expressive power, and the observation that the formulas in the five systems described here are all expressible as classical tenses does not imply that the language of classical tenses is itself a suitable tense logical system. Although we can express all

¹⁵There is a weaker sense in which U and S can be expressed with dating sentences. Let $U(\mathbf{i}, A, B)$ be $i \wedge \mathcal{F}(A \wedge \neg \mathcal{P} \neg (\mathcal{F}_i \vee i \vee B))$ and $S(\mathbf{i}, A, B)$ be $i \wedge \mathcal{P}(A \wedge \neg \mathcal{F}(\mathcal{P}_i \vee i \vee B))$. Then $U(\mathbf{i}, A, B)$ is satisfiable in Blackburn's system iff $U(A, B)$ is satisfiable in the since-until system and $S(\mathbf{i}, A, B)$ is satisfiable iff $S(A, B)$ is.

the classical tenses in English, it is not the tense mechanism that allows us to do so. English sentences like *For every instant t , if t succeeds t_0 there is an instant t' , such that t' succeeds t and t succeeds t_0 and John is asleep at t'* , however useful in explaining the meaning of first order formulas, are not the sort of sentences for which one would expect to find a phrase-by-phrase representatives in an idealized language isolating the tense-and-aspect features of English. One can object to Saarinen's **D**, Blackburn's dating sentences, Vlach's **B**, and Vlach and Cresswell's $\mathcal{I}_j$'s and $\mathcal{N}_j$'s on similar grounds. It is possible, of course, that some of these systems make particularly good intermediaries between tense constructions of natural language and truth conditions, or that there is some other sense in which they are especially suitable as tense logics for natural language, but such claims need arguments beyond demonstrations of expressive capacity. Indeed the fact that we can express very simply in these languages ideas that in English require complex constructions (perhaps involving quantifier phrases variable expressions) suggests that they are unsuitable on some conceptions of tense logic. On the other hand, if there are ideas we can express simply and uniformly in English, the mere observation that a tense-logical system has sufficient expressive power to somehow express them, may not be evidence in favor of the system. For example, the fact that prefixing a sufficiently long string of backspace operators to an embedded formula causes it to be evaluated at the moment of utterance does not mean that the backspace system is a good model of the English *now*.

Part of the difficulty in judging the adequacy of tense logical systems for natural language is discerning the linguistic data itself. It is not clear, for example, whether *John said he would come* does have the reading indicated in (54a) implying that he said he would come *by now*, or whether that inference, when legitimate, is based on (extralinguistic) contextual cues. Similarly, the observation that *Joe said that a child had been born who would become ruler of the world* is consistent with two possible utterances by Joe does not establish that (57) is ambiguous between (57a) and (57b). Saarinen maintains that a sentence of the form *A reported that B believed that C said John would go* has at least four readings, according to whether John's alleged departure occurs in the future of C's saying, B's believing, A's reporting, or the utterance time. Since the first of these readings is true if any of the others is, one can't expect to find a case which *requires* readings other than the first. The plausibility of there being such readings is undermined by observation that a similar ambiguity does not occur when the *would* is in the scope of future operators. *A will report (next week) that B said (the previous day) that C would go* is not made true by A's reporting next week that B said 'C went yesterday,' as it would if 'C would go' could refer to a time future to the utterance moment. While an adequate logic for the tenses of natural language may require greater 'time-tracking' capabilities than Priorean tense logic, there is not strong evidence for the

thesis that it be able to ‘remember’ at each stage in the evaluation times at which previous clauses were evaluated.

4.4 Galton’s logic of aspects and events

English discourse presumes a universe of events and states with internal structure as well as temporal location. The language of Priorean tense logic is built solely from formulas, boolean connectives, and operators of temporal location. It is reasonable to try to enrich the language so that more of the internal structure of events can be described. In recent years there has been a proliferation of work in this area motivated by concerns in deontic logic and action theory. (See, for example, Jones and Sergot [1996] and the references therein.) For the most part, however, that work has not focussed on temporal or natural language considerations. There is a large and growing semantics literature on events and aspect, but much of it is too detailed to be considered part of a ‘logic’ of tense. In this section we sketch some ideas in the spirit of Galton [1984; 1987a], which seem to strike a good balance between simplicity and fidelity to ‘surface’ phenomena of English.

The idea that sentences in the future and present perfect can be represented by attaching $\mathcal{F}$ and $\mathcal{P}$ to some more basic sentence is plausible for sentences describing states but not those describing events. *The cat has been on the mat* is true now if *the cat is on the mat* was true before, but to say that *John has built a house* is true now if *John builds a house* was true before is confusing, since we don’t normally use the present tense to indicate that an event is true at the present time. (Indeed, since events like house building occur over extended intervals, it is not clear what the ‘present’ time would be in this case.) Let us instead add a class of *event letters* $E_1, E_2, \dots$ to the language along with two event-to-formula *e-f aspect operators* $\mathcal{P}erf$ and $\mathcal{P}ros$, which attach to event letters to produce formulas.¹⁶ (The tense operators $\mathcal{P}$ and $\mathcal{F}$ and the boolean operators, as usual, apply to formulas to form formulas.) Let us provisionally say that an interpretation assigns to each event letter a set $I(E)$ of *occurrence intervals*. $\mathcal{P}rosE$ is true at t if t precedes (all of) some interval in $I(E)$; $\mathcal{P}erfE$, if t succeeds (all of) some interval in $I(E)$. One may wonder what hinges on the distinction between an event’s occurring at a time and a formula’s being true at a time. Granting that we don’t normally say that *John builds a house* is true, say, in the spring of 1995, we might find it convenient to stipulate that it be true then if one of John’s house buildings occurs at that time. One advantage of not doing so is that the event/formula distinction

¹⁶Galton uses the label ‘imperfective’ in place of ‘e-f’, and the label ‘perfective’ in place of our *f-e*.

provides a sorting that blocks inappropriate iterations of aspect operators. Another is that the distinction makes it possible to retain the Priorean notion that all formulas are to be evaluated at instants even when the events they describe occupy extended intervals. The tense logical systems that result from this language so interpreted will contain the usual tense logical principles, like $\mathcal{F}A \rightarrow \mathcal{F}\mathcal{P}A$ as well as event analogs of some of these, like $\mathcal{P}rosE \rightarrow \mathcal{F}\mathcal{P}erfE$. Some tense theorems lack event analogs. For example, $\mathcal{F}\mathcal{P}A \rightarrow (\mathcal{F}A \vee \mathcal{P}A \vee A)$ is valid when time does not branch towards the past, but $\mathcal{F}\mathcal{P}erfE \rightarrow \mathcal{P}erfE \vee \mathcal{P}rosE$ is not (because E may occur only at intervals containing the present moment).

We may add to this logic another *e-f* operator $\mathcal{P}rog$ such that $\mathcal{P}rogE$ is true at t iff t belongs to an interval at which E occurs. Thus $\mathcal{P}rogE$ asserts that event E is in progress. In view of the discussion in Section 3.4 above, it should be obvious that $\mathcal{P}rog$ is a poor representation of the English progressive. It can perhaps be viewed as an idealization of that construction which comes as close to its meaning as is possible with a purely temporal truth condition. (Analogous justifications are sometimes given for claims that the material conditional represents the English ‘if...then’ construction.) The new connective allows us to express the principle that eluded us above: $\mathcal{F}\mathcal{P}erfE \rightarrow \mathcal{P}erfE \vee \mathcal{P}rosE \vee \mathcal{P}rogE$.

Since Zeno of Elea posed his famous paradoxes in the fifth century *BC*, accounts of events and time have been tested by a number of puzzles. One Zeno-like puzzle, discussed in Hamblin [1971; 1971a], Humberstone, and Galton [Humberstone, 1979; Galton, 1984; Galton, 1987a], is expressed by the following question. ‘At the instant a car starts to move, is it moving or at rest?’ To choose one alternative would seem to distort the meaning of *starting to move*, to choose both or neither would seem to violate the laws of non-contradiction or excluded middle. Such considerations lead Galton to a slightly more complicated interpretation for event logic. Events are not assigned sets of occurrence intervals, but rather sets of interval pairs (B, A) , where B and A represent the times before the event and the times after the event (so that, if time is linear, B and A are disjoint initial and final segments of the set of times.) The clauses in the truth definition are modified appropriately. For example, $\mathcal{P}rogE$ is true at t if, for some $(B, A) \in I(E)$, $t \in -B \cap -A$, where $-A$ and $-B$ are those times of the model that do not belong to A and B . $\mathcal{P}erfE$ is true at t if, for some $(B, A) \in I(E)$, $t \in B$. For an event like the car’s starting to move, any (B, A) in the occurrence set will be exhaustive, i.e., $B \cup A$ will contain all times. Such events are said to be *punctual* (although we must distinguish these from events that occupy a ‘point’ in the sense that $B \cup A$ always omits a single time). A punctual event does not really occur ‘at’ a time, nor is it ever in the process of occurring. Instead, it marks a boundary between two states, like the states of rest and motion. When E is punctual, $\mathcal{P}rogE$ is always false, and so the principle $\mathcal{F}\mathcal{P}erfE \rightarrow \mathcal{P}erfE \vee \mathcal{P}rosE \vee \mathcal{P}rogE$

reduces to $\mathcal{F}\mathcal{P}erfE \rightarrow \mathcal{P}erfE \vee \mathcal{P}rosE$, which we have observed not to be valid without the stipulation that E is punctual.

We may also wish to add f - e aspect operators that apply to formulas to form event-expressions. Galton suggests the ‘ingressive’ operator $\mathcal{I}ngr$ and the ‘pofective’ operator $\mathcal{P}o$, where, for any formula A , $\mathcal{I}ngrA$ is the event of A ’s *beginning* to be true, and $\mathcal{P}oA$ is the event (or state) of A ’s being *true for a time*. In the ‘before-after’ semantics, these operators can be interpreted by the clauses below:

$$I(\mathcal{I}ngrA) = \{(B, -B) : A \text{ is true throughout a non-empty initial segment of } -B, \text{ and false throughout a nonempty final segment of } B\}$$

$$I(\mathcal{P}oA) = \{(B, C) : -B \cap -C \text{ is not empty, } A \text{ is true throughout } -B \cap -C \text{ and } A \text{ is false at some point in every interval that properly contains } -B \cap -C\}$$

Thus, $\mathcal{I}ngrA$ is always punctual, and $\mathcal{P}oA$ is never punctual. Notice that $-B \cap -C$ can be a singleton, so that being true for ‘a time,’ on this interpretation, includes being true for an instant. We get principles like $\mathcal{P}ros \mathcal{I}ngrA \rightarrow (\neg A \wedge \mathcal{F}A) \wedge \mathcal{F}(\neg A \wedge \mathcal{F}A)$, $\mathcal{P}erf \mathcal{I}ngr \mathcal{P}erfE \rightarrow \mathcal{P}erfE$, and $\mathcal{P}rog \mathcal{P}oA \rightarrow \mathcal{P}\neg A \wedge A \wedge \mathcal{F}\neg A$. It is instructive to consider the converses of these principles. If A is true and false at everywhere dense subsets of the times (for example if time is the reals and A is false at all rationals and true at all irrationals), then at the times A is false $\neg A \wedge \mathcal{F}A$ is true, but $\mathcal{I}ngrA$ has no occurrence pairs, and so $\mathcal{P}ros \mathcal{I}ngrA$ is false. Thus the converse of the first principle fails. Likewise, if E occurs repeatedly throughout the past (for example, if time is the reals and $I(E) = \{(-\infty, n][n+1, \infty)\}$) then $\mathcal{P}erfE$ is true at all times, which implies that $\mathcal{I}ngr \mathcal{P}erfE$ has an empty occurrence set, $\mathcal{P}erf \mathcal{I}ngr \mathcal{P}erfE$ is everywhere false, and the converse to the second principle fails. The converse to the third principle is valid, for if $\mathcal{P}\neg A \wedge A \wedge \mathcal{F}\neg A$ is true at t , then, letting $\cap S$ be the intersection of all intervals S such that $t \in S$ and A is true throughout S , the occurrence set of $\mathcal{P}oA$ includes the pair $(\{x : \forall y \in \cap S, x < y\}, \{x : \forall y \in \cap S, x > y\})$ and $\mathcal{P}rog \mathcal{P}oA$ is true at t . (The principle would fail, however, if we took $\mathcal{P}oA$ to require that A be true throughout an *extended* period.) As a final exercise in Galtonian event logic, we observe that it provides a relatively straightforward expression of Dedekind continuity (see Burgess [2001]). The formula $\mathcal{P}erf \mathcal{I}ngr \mathcal{P}erfE \rightarrow \mathcal{P}(\mathcal{P}erfE \wedge \neg \mathcal{P} \mathcal{P}erfE) \vee \mathcal{P}(\neg \mathcal{P}erfE \wedge \neg \mathcal{F} \neg \mathcal{P}erfE)$ states that, if there was a cut between times at which $\mathcal{P}erfE$ was false and times at which it was true, then either there was a first time when it was true or a last time when it was false. It corresponds to Dedekind continuity in the sense that a dense frame verifies the formula if and only if the frame is Dedekind continuous.

The view represented by the ‘before-after’ semantics suggests that events of the form *IngrA* and other punctual events are never in the process of occurring, but somehow occur ‘between’ times. However plausible as a metaphysical theory, this idea seems not to be reflected in ordinary language. We sometimes accept as true sentences like *the car is starting to move*, which would seem to be of the form *ProgIngrA*. To accommodate these ordinary-language intuitions, we might wish to revert to the simpler occurrence-set semantics. *IngrA* can be assigned short intervals, each consisting of an initial segment during which *A* is false and a final segment at which *A* is true. On this view, *IngrA* exhibits *vagueness*. In a particular context, the length of the interval (or a range of permissible lengths) is understood. When the driver engages the gear as the car starts to move he invokes one standard, when the engineer starts the timer as the car starts to move she invokes a stricter one. As in Galton’s account, the Zeno-like puzzle is dissolved by denying that there is an instant at which the car starts to move. The modified account concedes, however, that there are instants at which the car is starting to move while moving and other instants at which it is starting to move while not moving.

Leaving aside particular issues like the semantics of punctual events and the distinction between event-letters and sentence-letters, Galton’s framework suggests general tense-logical questions. The *f–e* aspect operators, like *Ingr* and *Po* can be viewed as operations transforming instant-evaluated expressions into interval-evaluated (or interval-occupying?) expressions, and the *e–f* aspect operators, like *Perf* and *Prog*, as operations of the opposite kind. We might say that traditional tense logic has investigated general questions about instant/instant operations and that interval tense logic has investigated general questions about operations taking intervals (or pairs of intervals) to intervals. A general logic of aspect would investigate questions about operations between instants and intervals. Which such operations can be defined with particular metalinguistic resources? Is there anything logically special about those (or the set of all those) that approximate aspects of natural language? The logic of events and aspect would seem to be a fertile ground for further investigation.

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